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QUANTUM CORRELATIONS AND QUANTUM COHERENCE IN OPEN QUANTUM SYSTEMS

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A mi vida y a mi madre.

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"The scientist does not study nature because it is useful,
he studies it because he delights in it,
and he delights in it because it is beautiful.
If nature were not beautiful, it would not be worth knowing
and if nature were not worth knowing,
life would not be worth living".
(Henri Poincare)

ABSTRACT

In this thesis, methods for the study of open quantum systems are developed and different aspects of Markovian and non-Markovian dynamics are analyzed. The first part of the thesis is devoted to the theoretical foundations of open quantum systems and a brief review of quantum correlations, emphasizing geometric quantum correlations. In the second part of the thesis, we present our main results. More specifically, we firstly provide analytical expressions for classical and total trace-norm (Schatten 1-norm) geometric correlations in the case of two-qubit X states. As an application, we consider the open-system dynamical behavior of such correlations under phase and generalized amplitude damping evolutions. Then, we show that geometric classical correlations can characterize the emergence of the pointer basis of an apparatus subject to decoherence in either Markovian or non-Markovian regimes. Secondly, we provide a characterization of memory effects in non-Markovian system-bath interactions from a quantum information perspective. In particular, we establish sufficient conditions for which generalized measures of multipartite quantum, classical, and total correlations can be used to quantify the degree of non-Markovianity of a local quantum decohering process. We illustrate our results by considering the dynamical behavior of the trace-distance correlations in multi-qubit systems under local dephasing and generalized amplitude damping. Finally, we investigate quantum coherence, discussing its connections with quantum correlations measurements and proposing it as a quantifier of non-Markovianity. As an example, the coherence of a qubit under non-Markovian amplitude damping is analytically discussed.

Keywords: Decoherence, open systems, quantum statistical methods, quantum information, entanglement and quantum nonlocality, mechanical instruments and equipment.

RESUMO

Nesta tese, são desenvolvidos métodos para o estudo de sistemas quânticos abertos e diferentes aspetos da dinâmica Markoviana e não-Markoviana são analisados. A primeira parte da tese é dedicada aos fundamentos teóricos dos sistemas quânticos abertos e uma breve revisão das correlações quânticas, focando nas correlações quânticas geométricas. Na segunda parte da tese apresentamos nossos principais resultados. Em primeiro lugar, fornecemos expressões analíticas para a correlação geométrica clássica via norma-1 de Schatten no caso de estados X de dois qubit. Como aplicação, consideramos o comportamento dinâmico de um sistema aberto, de tais correlações sob a atuação de ruídos de atenuação de fase e atenuação de amplitude generalizada. Depois, mostramos que as correlações clássicas geométricas podem caracterizar o surgimento da base do ponteiro de um aparelho sujeito à decoerência nos regimes Markoviano ou não-Markoviano. Em segundo lugar, fornecemos uma caracterização de efeitos de memória em interações não-Markovianas do sistema com o banho desde uma perspetiva da informação quântica. Em particular, estabelecemos suficientes condições para as quais medidas multipartidas generalizadas de correlações quânticas, clássicas e totais podem ser usadas para quantificar o grau de não-Markovianidade de um processo local de decoerência quântica. Ilustramos os nossos resultados considerando o comportamento dinâmico das correlações traço-distância em sistemas multiqubit sob ruídos de atenuação de fase e atenuação de amplitude generalizada. Por último, investigamos a coerência quântica discutindo suas conexões com as medidas de correlações quânticas e propondo-a como um quantificador da não-Markovianidade. Como um exemplo, discutimos a coerência de um qubit sob o canal não-Markoviano de atenuação de amplitude.

Palavras-chave: Decoerência, sistemas abertos, métodos de estatística quântica, informação quântica, emaranhamento e não-localidade quântica, instrumentos e equipamentos mecânicos.

RESUMEN

En esta tesis, se desarrollan métodos para el estudio de sistemas cuánticos abiertos y se analizan diferentes aspectos de la dinámica Markoviana y no Markoviana. En la primera parte de la tesis se presenta una revisión de los fundamentos teóricos de los sistemas cuánticos abiertos y una breve reseña de las correlaciones cuánticas, enfatizando en las correlaciones cuánticas geométricas. En la segunda parte de la tesis presentamos nuestros principales resultados. En primer lugar proporcionamos expresiones analíticas para correlaciones geométricas clásicas y totales a través de la norma 1 de Schatten para el caso de estados X de dos qubits. Como aplicación, consideramos el comportamiento dinámico del sistema abierto de tales correlaciones para el ruido de disipación de la amplitud y el canal de defasamiento. Luego, mostramos que las correlaciones geométricas clásicas pueden caracterizar el surgimiento de la base de puntero de un aparato sujeto a decoherencia en regímenes Markovianos o no Markovianos. En segundo lugar, desde una perspectiva de información cuántica proporcionamos una caracterización de los efectos de memoria en interacciones no Markovianas entre el sistema y el baño. En particular, establecemos condiciones suficientes para las cuales se pueden usar medidas generalizadas de correlaciones cuánticas, clásicas y totales multipartitas para cuantificar el grado de no Markovianidad para un canal de desfasamiento local. Ilustramos nuestros resultados considerando el comportamiento dinámico de las correlaciones traza-distancia en sistemas multi-qubit bajo un ruido de desfasamiento local y un ruido de disipación de la amplitud generalizado. Finalmente, investigamos la coherencia cuántica, discutiendo sus conexiones con las medidas de correlaciones cuánticas y proponiéndola como un cuantificador de no Markovianianidad. Como un ejemplo, se discute analíticamente la coherencia de un qubit para el caso de un canal no Markoviano de disipación de la amplitud.

Palabras clave: Decoherencia, sistemas abiertos, métodos de estadística cuántica, información cuántica, entrelazamiento y no-localidad cuántica, instrumentos mecánicos y equipos.

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LIST OF ABBREVIATIONS AND ACRONYMS

AD	Amplitude damping
BF	Bit-flip
BPF	Bit-phase-flip
BLP	Breuer, Laine and Piilo
CP	Completely positive
CPTP	Completely positive and trace-preserving
EOF	Entanglement of formation
HS	Hilbert-Schmidt
ICPTP	Incoherent completely positive and trace-preserving
LOCC	Local operations and classical communications
GAD	Generalized amplitude damping
GQD	Geometric quantum discord
PD	Phase Damping
PF	Phase-flip
RHP	Rivas, Huelga and Plenio
TCL	Time-convolutionless
TND	Trace-norm of discord

CONTENTS

1 INTRODUCTION

Recent developments about the problem of decoherence in quantum mechanics have heightened the need for the consistent description of open quantum systems (NIELSEN; CHUANG, 2000; BREUER; PETRUCCIONE, 2002; RIVAS; HUELGA, 2012). The theory of open quantum systems is fundamental to describe a wide range of applications of quantum physics, since perfect isolation of quantum systems is not possible and since a complete microscopic description or control of the environmental degrees of freedom is not feasible or only partially so (BREUER; PETRUCCIONE, 2002). One of the greatest challenges faced by many experiments is the inevitable interaction of the quantum system with its surroundings, this interaction generates system-environment correlations which induce loss of quantum coherence and dissipation (BREUER et al., 2016). Realistic quantum mechanical systems are thus open systems governed by a non-unitary time development which describes all features of irreversible dynamics such as the dissipation of energy, the relaxation to a thermal equilibrium or a stationary nonequilibrium state and the decay of quantum coherences and correlations (ALICKI; LENDI, 1987; BREUER; PETRUCCIONE, 2002; LINDBLAD, 1976).

The main goal of the theory of open quantum systems is to avoid having to integrate the full system, comprising both the open quantum system itself and its environment, by describing the dynamics of the open system in its reduced Hilbert space (RIVAS; HUELGA, 2012; BREUER; PETRUCCIONE, 2002; NIELSEN; CHUANG, 2000). Namely, the open systems are treated with the usual laws of dynamics, by regarding them as subsystems of larger systems which are closed (i.e. which obey the usual laws of dynamics with a well-defined Hamiltonian). The structure of the system-environment initial state is fundamental to determine the evolution for the reduced density matrix of the open quantum system, defined by tracing out the environment degrees of freedom from the full system density matrix. Under certain conditions, there is a well-established treatment of the dynamics of open quantum systems in which the open system's time evolution is represented by a dynamical semigroup (BREUER; PETRUCCIONE, 2002; LINDBLAD, 1976; GORINI et al., 1978; INGARDEN; KOSSAKOWSKI, 1975; KOSSAKOWSKI, 1972). The importance of the dynamical semigroup concept is that it generalizes the evolution operator to open systems. The notion of the quantum dynamical semigroup is defined using the concept of a completely positive map. Moreover, the Lindblad formalism (LINDBLAD, 1976; GORINI et al., 1978; KOSSAKOWSKI, 1972) replaces description in terms of the Hamiltonian operator of the system, for a completely positive dynamical semigroup with bounded generators. This formalism leads to a Markovian quantum master equation, which is obtained if one assumes a weak coupling system-environment interaction and a

memoryless environment. Hence, the environment recovers instantly from the interaction, leading to a continuous flow of information from the system to the environment (VEGA; ALONSO, 2017).

However, recent developments in the field of new materials have led to observe and control quantum systems at different times, length scales and energy ranges (BREUER et al., 2016; VEGA; ALONSO, 2017). In many of these scenarios, a large separation between system and environment time scales can no longer be assumed, leading to non-Markovian behavior and eventually a back-flow of information from the environment into the system. Indeed, the Markovian behavior is always an idealization in the description of the quantum dynamics, with non-Markovianity being non-negligible in a number of different scenarios, such as biological (ISHIZAKI; FLEMING, 2009; REBENTROST; CHAKRABORTY; ASPURU-GUZIK, 2009; LIANG, 2010; CHEN et al., 2015) or condensed matter systems (WOLF et al., 2008; APOLLARO et al., 2011; HAIKKA; JOHNSON; MANISCALCO, 2013). A considerable amount of literature has been published on rigorously define non-Markovian dynamics in the quantum case, different approaches have been followed and several methods have been proposed (see, e. g., (RIVAS; HUELGA; PLENIO, 2014; POLLOCK et al., 2015; POLLOCK et al., 2018; BREUER et al., 2016; VEGA; ALONSO, 2017)). From an applied point of view, non-Markovian dynamics may be a resource for quantum tasks through an increase in the capacities of quantum channels (BYLICKA; CHRUŚCIŃSKI; MANISCALCO, 2014). Moreover, it also exhibits applications in fault-tolerant quantum computation (AHARONOV; KITAEV; PRESKILL, 2006). Basically, two main questions need to be discussed in this context (RIVAS; HUELGA; PLENIO, 2014):

- What is a quantum Markovian process and hence what are non-Markovian processes? (characterization problem).
- If a given process deviates from Markovianity, by how much does it deviate? (quantification problem).

In this thesis we examine both questions. Precisely, concerning the characterization problem, we adopt the so-called divisibility property of dynamical maps as a definition of quantum Markovian processes. Some techniques which allow for a systematic description of non-Markovian dynamics of open systems are showed in terms of Kakajima-Zwanzig projection operator method (NAKAJIMA, 1958; ZWANZIG, 1960; MORI, 1965). It is our prospect show a general framework for the study of open systems in terms of mathematical tools which will allow us to address the characterization of the non-Markovianity for the scope of this thesis. On the other hand, concerning the quantification problem, researches have recently shown an increased interest in defining the border between the regions of Markovian and non-Markovian quantum dynamics. The concept of non-

Markovian degree was proposed in (CHRUŚCIŃSKI; MANISCALCO, 2014) where is established a formal analogy between the entanglement theory and quantum evolutions. Other approaches are based on fidelity (RAJAGOPAL; DEVI; RENDELL, 2010), channel capacity (BYLICKA; CHRUŚCIŃSKI; MANISCALCO, 2014), geometry of the set of accessible states (LORENZO; PLASTINA; PATERNOSTRO, 2013) and distinguishability between quantum states (BREUER; LAINE; PIILO, 2009). For the purpose of this thesis, we focus on two of the most widely studied and significant measures of the degree of non-Markovianity. The first was proposed by Rivas, Huelga and Plenio (RHP measure) (RIVAS; HUELGA; PLENIO, 2010), where the structural properties of quantum channels are proposed to define non-Markovianity, here measuring violations of the completely positivity of the map. The second was proposed Breuer, Laine and Piilo (BLP measure) (BREUER et al., 2016; BREUER; LAINE; PIILO, 2009). In this approach the non-Markovianity based on the study of the dynamical behavior of the distinguishability of states and the quantum memory concept. This viewpoint directly addresses the issue of the experimental detection of quantum non-Markovian process, which can be obtained by suitable quantum tomographic measurements (BREUER et al., 2016; FANCHINI et al., 2014).

In recent years, there has been an increasing interest in investigating the dynamics of quantum correlations and quantum coherence in Markovian and non-Markovian open quantum systems by comparing the evolution of different types of correlations in specific models, typically two-qubits coupled to two local baths or one common bath (HU et al., 2017). Quantum correlations and quantum coherence are two fundamental concepts in quantum theory (NIELSEN; CHUANG, 2000; HU et al., 2017). While quantum correlations characterize the quantum features of a system with at least two parties, quantum coherence is defined already for a single system. Also, from a practical point of view, quantum correlations and quantum coherence play an important role as physical resources for various quantum information and computation tasks (MODI et al., 2012; STRELTSOV; ADESSO; PLENIO, 2017).

Since the early days of quantum information theory, entanglement has been viewed as the main feature that gives quantum computers an advantage over their classical counterparts (MODI et al., 2012; HORODECKI et al., 2009). The view that entanglement is crucial is also supported by foundational considerations, for it is known that Bell's inequalities require genuine entanglement to exceed the classically-determined limit for correlations (MODI et al., 2012). Schrödinger captured all this in (SCHRÖDINGER, 1935), saying entanglement is "not just one of many traits, but the characteristic trait of quantum physics". In parallel, a significant current discussion in quantum correlations is the concept of quantum discord. Since it was reportered in 2001, the study of quantum discord has attracted a lot of interest. Quantum discord was formulated by Henderson and Vedral (HENDERSON; VEDRAL, 2001) as well as by Ollivier and Zurek (OLLIVIER; ZUREK,

2001), the authors concluded that entanglement does not account for all non-classical correlations and that even separable states usually contain correlations that are not entirely classical. Along this line, a series of discord-like correlation measures were proposed, and studied from different aspects (DATTA, 2010; ADESSO et al., 2014; CIANCIARUSO et al., 2015; MODI et al., 2012).

At the present time, there are many ways of understanding the fact that classical correlations and entanglement do not exhaust all possible correlations in quantum systems (MODI et al., 2011; CéLERI; MAZIERO; SERRA, 2011; SARANDY; OLIVEIRA; AMICO, 2012). It is important to point out that quantum correlations appear as key signatures, with operational roles e.g. in quantum metrology (MODI et al., 2011; GIROLAMI; TUFARELLI; ADESSO, 2013; GIROLAMI et al., 2014), entanglement activation (STRELTSOV; KAMPERMANN; BRUß, 2011; PIANI et al., 2011), and information encoding and distribution (GU et al., 2012; STRELTSOV; ZUREK, 2013). The proposed discord-like quantum correlation measures can be defined through a number of distinct formulations, which are based on the relative entropy (MODI et al., 2010), Hilbert-Schmidt norm (DAKIĆ; VEDRAL; BRUKNER, 2010; BELLOMO et al., 2012), trace-norm (PAULA; OLIVEIRA; SARANDY, 2013; NAKANO; PIANI; ADESSO, 2013), or Bures norm (SPEHNER; ORSZAG, 2013; BROMLEY et al., 2014). All of these distinct versions of geometric quantum correlations can be generally described by a unified framework in terms of a distance (or pseudo distance) function (MODI et al., 2011; BRODUTCH; MODI, 2012; PAULA et al., 2014).

This thesis will examine quantum, classical, and total correlations by the trace norm, which corresponds to the Schatten 1-norm. Initial published studies were limited for the simple case of mixed two-qubit systems in Bell-diagonal states, were analytical expressions have been found for quantum, classical, and total correlations (PAULA et al., 2013b; AARONSON et al., 2013; PAULA et al., 2014). For the more general case of two-qubit X states, only the quantum contribution for the geometric correlation had been analytically derived (CICCARELLO; TUFARELLI; GIOVANNETTI, 2014). One of specific objectives of this thesis was to close this gap, providing closed analytical expressions for the classical and total correlations of arbitrary two-qubit X states. Remarkably, they are shown to be as simple to be computed as in the case of Bell-diagonal states. These results were published in (OBANDO; PAULA; SARANDY, 2015).

Correlations are closely related to the characterization of Markovian evolutions in open quantum systems. Rigorously, non-Markovianity can be defined through the deviation of a dynamical evolution map from a divisible completely positive trace-preserving map (BREUER; LAINE; PIILO, 2009). This behavior is manifested both in entanglement (RIVAS; HUELGA; PLENIO, 2010) and in other correlation sources (LUO; FU; SONG, 2012; HAIKKA; JOHNSON; MANISCALCO, 2013; FANCHINI et al., 2014; HU et al., 2012; DHAR; BERA; ADESSO, 2015), providing an approach that takes advantage of

quantum information tools in the open-systems realm. More specifically, non-Markovianity can be interpreted in this context as a flow of information back to the system due to its interaction with the environment, which may imply into a non-monotonic behavior of correlations as a function of time.

In this thesis we use the general framework in terms of a distance function with the aim of characterizing non-Markovianity through multipartite measures of quantum, classical, and total correlations. Our approach for non-Markovianity includes all these measures as particular cases and characterizes the non-Markovian behavior by taking into account the RHP mesure (RIVAS; HUELGA; PLENIO, 2010) and BLP mesure (BREUER et al., 2016; BREUER; LAINE; PIILO, 2009). Specifically, the objective of this study published in (PAULA; OBANDO; SARANDY, 2016) was to provide a rigorous description of the hypotheses over local dynamical maps under which quantum, classical, and total correlations can be used as measures of non-Markovianity.

Concerning quantum coherence, it has long been a topic of great interest in a wide range in fields of quantum information science and quantum technologies, such as quantum thermodynamics (GOUR et al., 2015; LOSTAGLIO; JENNINGS; RUDOLPH, 2015; LOSTAGLIO et al., 2015), reference frames (BARTLETT; RUDOLPH; SPEKKENS, 2007), quantum biology (LAMBERT et al., 2013), quantum transport (BRANDES, 2010; KIM; CHOI, 2005) and nanoscale physics (KARLSTRÖM et al., 2011; ENGEL et al., 2007). Inspired on the recent developments about the quantitative characterization of quantum coherence and its fundamental character (BAUMGRATZ; CRAMER; PLENIO, 2014; STRELTSOV; ADESSO; PLENIO, 2017; HU et al., 2017), one of the aims of this thesis is the study of quantum coherence and the investigation of the quantum coherence as a witness of non-Markovianity of an incoherent open system dynamics through the non-monotonic behavior of quantum coherence measures.

This thesis is organized as follows: Chapter two reviews the theory of open quantum systems in a Markovian limit. Section 2.1 gives a brief overview of the postulates of quantum mechanics. Section 2.3 establishes the context of the dynamics of open systems. Section 2.4 introduces the concept of quantum Markov process through quantum dynamical semigroup. This is crucial in order to understand why the divisibility property provides a good definition of quantum Markovianity. Section 2.5 reviews the formalism of Markovian master equations and explains why these quantum processes can be considered as memoryless. Section 2.6 shows some examples of noise in quantum systems.

Chapter three is devoted to study of non-Markovian open systems. This chapter is basically divided in two parts: the first part (section 3.1 and section 3.2) gives a mathematical framework for non-Markovian dynamics by projection operator techniques. Section 3.2 describe the non-Markovian setting used in the applications section of this thesis. The second part, section 3.3 is related to the BPL measure.

The fourth Chapter reviews the theory of quantum correlations. Section 4.2 is devoted to study of the quantum discord. Section 4.2 is related with the Geometric quantum correlation measures. Section 4.4 discuss in a general framework the criteria for a proper measure of correlation.

Chapter five present our original results about quantum correlations for X states. Section 5.1 describes the characterization of X states. Section 5.2 shows one of our main results of this thesis, the analytical expression for geometrical classical and total correlations for two-qubits X states. The analytical expressions for the classical correlation of X states can be applied as a powerful resource to characterize the open-system dynamics in rather general environments. Therefore, in section 5.3 we illustrate the application in a process known like emergence of the pointer basis (ZUREK, 2003). This is exploited in a general scenario of X states, for either Markovian and non-Markovian evolutions.

Chapter six is concerned with ours results about the characterization of the memory effects in non-Markovian systems. Section 6.1 reviews briefly the characterization of non-Markovianity. Section 6.2 present our main results about the characterization of non-Markovianity in a general framework by multipartite measures of quantum, classical and total correlations. Section 6.3 illustrate some applications considering the dynamics of the trace-distance correlations in qubit systems under either local dephasing or generalized amplitude damping.

Chapter seven shows our results about quantum coherence. Section 7.1 reviews the quantum coherence measurement, emphasize in the quantum coherence for one-qubit and basis independent definitions. Section 7.2 discusses the connections of quantum coherence with quantum correlations measurements. Section 7.5 discuss the quantifying non-Markovinity by quantum coherence.

Finally, Chapter eight is devoted to conclusions of our thesis and to outline some problems which remain open and possible future research.

2 OPEN QUANTUM SYSTEMS

The aim of this chapter is to review some of the most relevant concepts of the theory of open quantum systems which will be employed throughout the entire thesis. Roughly speaking, the perfect isolation of a particular quantum system from its surrounding is impossible (BREUER; PETRUCCIONE, 2002). In reality, quantum systems are influenced by its surroundings. In this sense, the theory of open quantum systems is useful for describing quantum systems where the influence of the environment cannot be neglected. Unlike the case of closed systems, system-environment interactions cannot be represented in terms of a unitary time evolution. The system state is properly described by the density matrix formalism (see section 2.1). In the simplest case of the dynamics of an open system, the density matrix evolution is given by a quantum memoryless Markov master equation whose unitary part contains the dynamics which are given by the system Hamiltonian and the effect of the environment on the system is described by the non-unitary dissipator.

This chapter is organized as follows. Section 2.1 describes the postulates of quantum mechanics. Section 2.2 introduce briefly the quantum qubit concept. Section 2.3 presents a brief discussion about quantum open systems. Section 2.4 introduces the concepts of a quantum dynamical semigroup and some basic facts of quantum memoryless Markovian open systems. Section 2.5 presents the theory of quantum master equations in the Markovian regime. Finally, section 2.6 presents a discussion about quantum noise.

2.1 Postulates of quantum mechanics

In order to discuss the dynamic of open quantum systems, in this section we shall first briefly discuss the postulates of quantum mechanics that let describe the behavior of quantum systems (BALLENTINE, 1998; NIELSEN; CHUANG, 2000; BLUM, 2012).

Postulate 1. *Associated to any physical system is a Hilbert space* $\mathcal{H}$[1] *of some dimension* D *known as the state space of the system. The system is completely described by its density operator, which is a positive operator* ρ *with trace one, acting on the state space of the system.*

Thereby, we consider a mixture of independently prepared states $|\psi_n\rangle$ $(n = 1, 2, ...)$ with statistical weights W_n. The density operator describing the mixture is then defined

[1] Complete and complex vector space with inner product.

as (NIELSEN; CHUANG, 2000; BLUM, 2012):

$$\rho = \sum_n W_n |\psi_n\rangle\langle\psi_n|, \tag{2.1}$$

where the sum extends over all states present in the mixture. The operator Eq. (2.1) can be expressed in matrix form in the $\{|\phi_n\rangle\}$ representation[2] by (BLUM, 2012):

$$\rho = \sum_{nm'm} W_n a_{m'}^{(n)} a_m^{(n)*} |\phi_{m'}\rangle\langle\phi_m|, \tag{2.2}$$

where $|\psi_n\rangle = \sum_{m'} a_{m'}^{(n)} |\phi_{m'}\rangle$. The terms density operator and density matrix have essentially the same meaning and are used interchangeably. A density operator satisfies the following properties (BLUM, 2012):

- ρ is Hermitian, that is, the density matrix satisfies the condition:

$$\langle\phi_i|\rho|\phi_j\rangle = \langle\phi_j|\rho|\phi_i\rangle^*. \tag{2.3}$$

- Since, the probability of finding the system in the state $|\psi_n\rangle$ is W_n and the probability that $|\psi_n\rangle$ can be found in the state $|\phi_m\rangle$ is $|a_m^{(n)}|^2$, the probability of finding the system in the state $|\phi_m\rangle$ is given by the diagonal element:

$$\rho_{mm} = \sum_n W_n |a_m^{(n)}|^2. \tag{2.4}$$

  This relation gives a physical interpretation of the diagonal elements of ρ. They are the total probabilities of measurement outcomes, taking into account the composition of the given mixture.

- ρ is positive semidefinite, this is $\langle\phi|\rho|\phi\rangle \geq 0$ for any non-zero state $|\phi\rangle$.

- The trace of ρ is one ($\mathrm{tr}\rho = 1$) independent of the representation.

- The expectation value of any operator $\mathcal{Q}$ is given by:

$$\langle\mathcal{Q}\rangle = \mathrm{tr}(\rho\mathcal{Q}). \tag{2.5}$$

[2] The particular states $|\phi_n\rangle$ are termed basis states and the state $|\psi\rangle = \sum_n a_n|\phi_n\rangle$ is said to be written in the $\{|\phi_n\rangle\}$ representation. We always assume that the basis states are orthonormal:

$$\langle\phi_n|\phi_m\rangle = \delta_{nm},$$

and complete:

$$\sum_n |\phi_n\rangle\langle\phi_n| = 1.$$

This relation is an important result. Since the expectation value of any operator can be obtained by use this equation, the density matrix contains all physically significant information on the system.

- A quantum system whose state $|\phi\rangle$ is known exactly is said to be in a pure state. The density operator for a pure state is given by:

$$\rho = |\psi\rangle\langle\psi|. \tag{2.6}$$

Otherwise, ρ is in a mixed state; it is said to be a mixture of the different pure states in the ensemble for ρ (NIELSEN; CHUANG, 2000). The sufficient and necessary condition that a given density matrix describes a pure state (BLUM, 2012) is:

$$\mathrm{tr}(\rho^2) = (\mathrm{tr}\rho)^2 = 1, \tag{2.7}$$

while a mixed state satisfies

$$\mathrm{tr}(\rho^2) < 1. \tag{2.8}$$

Postulate 2. *The evolution of density operator ρ is given by the von Neumann equation (also known as the Liouville–von Neumann equation) (BLUM, 2012).*

The evolution of quantum mechanical states is governed by Schrödinger's equation:

$$i\hbar\frac{\partial|\psi(t)\rangle}{\partial t} = H(t)|\psi(t)\rangle, \tag{2.9}$$

where the Hamiltonian H depends explicitly on the time, $\hbar$ is the Planck's constant, for the remained of this thesis we will assume that the factor $\hbar$ is absorbed into H, setting $\hbar = 1$. In this equation the solutions are not obtained by a simple way. Though, the solution may be generalized by introducing the time evolution operator $U(t)$:

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle, \tag{2.10}$$

by substitution of this equation into the Schrödinger equation (2.9) we obtain:

$$i\frac{\partial U|\psi(0)\rangle}{\partial t} = H(t)U|\psi(0)\rangle, \tag{2.11}$$

since this equation holds for any state $|\psi(0)\rangle$ this condition can be written as an operator equation:

$$i\frac{\partial U(t)}{\partial t} = H(t)U(t). \tag{2.12}$$

In order to ensure that the system at time $t = 0$ is in the state $|\psi(0)\rangle$, it is necessary to impose the initial condition $U(0) = \mathbb{I}$, where $\mathbb{I}$ is the identity operator. Consequently $U(0)^\dagger U(0) = \mathbb{I}$. For these conditions follows that $U^\dagger U$ must be the identity operator. Remarkably, if H is time independent we obtain by formally integrating Eq. (2.11):

$$U(t) = e^{-iHt}. \tag{2.13}$$

This operator contains all information on the time evolution of any state $|\psi(t)\rangle$ and hence also on the dynamics of the system. Let us now consider that at the time $t = 0$ a certain mixture is represented by the density operator (BLUM, 2012):

$$\rho(0) = \sum_n W_n |\psi_n(0)\rangle\langle\psi_n(0)|. \tag{2.14}$$

The states $|\psi_n(0)\rangle$ vary in time according to Eq. (2.13) and, therefore the density operator becomes a function of time:

$$\rho(t) = U(t)\rho(0)U(t)^\dagger, \tag{2.15}$$

differentiating this equation with respect to t and applying Eqs. (2.12), we obtain:

$$i\frac{\partial\rho(t)}{\partial t} = [H(t), \rho(t)], \tag{2.16}$$

with the commutator:

$$[H(t), \rho(t)] = H(t)\rho(t) - \rho(t)H(t).$$

This differential equation is called the Liouville equation. The time evolution of a density operator can be determined either from Eq. (2.15) or equivalently, from Eq. (2.16).

Postulate 3. *Quantum measurements are described by a collection $\{M_m\}$ of measurement operators. If the state of the quantum system is ρ immediately before the measurement then the probability that result m occurs is given by (BLUM, 2012; NIELSEN; CHUANG, 2000):*

$$p(m) = \mathrm{tr}(M_m^\dagger M_m \rho), \tag{2.17}$$

and the state of the system after the measurement is:

$$\frac{M_m \rho M_m^\dagger}{\mathrm{tr}(M_m^\dagger M_m \rho)}. \tag{2.18}$$

The measurement operator satisfy the completeness equation $\sum M_m^\dagger M_m = \mathbb{I}$.

Postulate 4. *The state space of a composite physical system $\mathcal{H}$ is the tensor product of the state space $\{\mathcal{H}_i\}$ of the component physical systems (BLUM, 2012). If we have systems labeled from 1 to n and system i is prepared in the state ρ_i, the state of the total system is $\rho_1 \otimes ... \otimes \rho_n$.*

The composite system $\mathcal{H}_A \otimes \mathcal{H}_B$ is represented by the density operator ρ_{AB}. The state of system A when considered alone is given by its reduce density operator $\rho_A = \mathrm{tr}_B \rho_{AB}$, where the partial trace operator tr_B is defined by:

$$\mathrm{tr}_{\mathrm{B}}(R \otimes S) = (\mathbb{I} \otimes \mathrm{tr})(R \otimes S) = R\mathrm{tr}(S),\ \forall R \in \mathcal{H}_A,\ \forall S \in \mathcal{H}_B. \tag{2.19}$$

2.2 Quantum bits

Let us focus on the general density matrix for the case of a qubit. The unit of quantum information is called the "quantum bit" or qubit in analogy with the classical binary bit, which is the indivisible unit of classical information (NIELSEN; CHUANG, 2000; PRESKILL, 1998). The state of a qubit is a vector in a two-dimensional Hilbert space, two possible states for a qubit are the states $|0\rangle$ and $|1\rangle$. These states form an orthonormal basis $\{|0\rangle, |1\rangle\}$, also known as computational basis. Therefore, the most general normalized state for a qubit is given by (NIELSEN; CHUANG, 2000; PRESKILL, 1998):

$$|\psi\rangle = a|0\rangle + b|1\rangle, \tag{2.20}$$

where a, b are complex numbers with $|a|^2 + |b|^2 = 1$. Physically speaking, a qubit corresponds typically to the two levels of some microscopic system such as a polarized photon, a trapped ion, a nuclear spin, etc (PRESKILL, 1998; NIELSEN; CHUANG, 2000; ORSZAG, 2007). In this way, Eq. (2.20) can be interpreted as general spin state of an object with spin-1/2. Then $|0\rangle$ and $|1\rangle$ are the spin up $|\uparrow\rangle$ and spin down $|\downarrow\rangle$ states along a particular axis such as the z-axis [3]. The state of a spin pointing in the (θ, ϕ) direction is given by:

$$|\psi(\theta, \phi)\rangle = \begin{pmatrix} e^{-i\phi/2} \cos\frac{\theta}{2} \\ e^{+i\phi/2} \sin\frac{\theta}{2} \end{pmatrix}. \tag{2.21}$$

Considering the general density matrix for the case of a qubit. The most general self-adjoint 2×2 matrix has four real parameters, and can be expanded in the basis $\{\mathbb{I}, \sigma_x, \sigma_y, \sigma_z\}$, where the corresponding Pauli matrices are given by:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.22}$$

Consequently, it is possible to write ρ as follows (PRESKILL, 1998; NIELSEN; CHUANG, 2000; BLUM, 2012):

[3] In the Pauli representation the state vectors are represented by two-dimensional column vectors: $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$$\begin{aligned}\rho(\vec{P}) &= \frac{1}{2}(\mathbb{I} + \vec{P}\cdot\vec{\sigma}),\\ &\equiv \frac{1}{2}(\mathbb{I} + P_x\sigma_x + P_y\sigma_y + P_z\sigma_z),\\ &= \frac{1}{2}\begin{pmatrix} 1+P_z & P_x - iP_y \\ P_x + iP_y & 1-P_z \end{pmatrix}.\end{aligned} \tag{2.23}$$

The components of $\{P_i\}$ define the Bloch vector. Which represents a vector that satisfies $|\vec{P}| \leq 1$, this is a necessary and sufficient condition for ρ to have non-negative eigenvalues. The density matrix Eq. (2.23) describes statistical mixture, while a Bloch vector satisfying $|\vec{P}| = 1$ represents a pure state. There exist a one-to-one correspondence between the possible density matrices of a single qubit and the points on the unit 3-ball $0 \leq |\vec{P}| \leq 1$, known as the Bloch sphere. For the case of a single qubit Eq. (2.21), we obtain the pure state:

$$\begin{aligned}\rho(\theta,\phi) &= |\phi(\theta,\phi)\rangle\langle(\theta,\phi)|,\\ &= \frac{1}{2}(\mathbb{I} + \hat{n}\cdot\vec{\sigma}),\end{aligned} \tag{2.24}$$

where $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. From the property $\frac{1}{2}\mathrm{tr}(\sigma_i\sigma_j) = \delta_{ij}$, we obtain:

$$\langle \hat{n}\cdot\vec{\sigma}\rangle_{\vec{P}} = \mathrm{tr}(\hat{n}\cdot\rho(\vec{P})) = \hat{n}\cdot\vec{P}. \tag{2.25}$$

Hence the vector $\vec{P}$ in Eq. (2.23) parametrizes the polarization of the spin. It is possible to determine the complete density matrix $\rho(\vec{P})$ by measuring $\langle\hat{n}\cdot\vec{\sigma}\rangle$ along each of three linearly independent axes.

2.3 Dynamics of open systems

Closed quantum systems are ideal quantum systems which are perfectly isolated from the environment. These states follow a unitary dynamics given by Eq. (2.16). Conversely, an open system is a quantum system S which is coupled to another quantum system E called the environment, where the open system dynamics is non-unitary. The combined total system $S + E$ generally is assumed closed. The interaction with the surroundings and the internal dynamics cause the change of the state of the subsystem S (BREUER; PETRUCCIONE, 2002). The Hilbert space of the total system $S + E$ is given by the tensor product space

$$\mathcal{H}_{SE} = \mathcal{H}_S \otimes \mathcal{H}_E, \tag{2.26}$$

where $\mathcal{H}_S$ and $\mathcal{H}_E$ denote the spaces of the S and E, respectively. The states of the total system ρ_{SE} are physical states that preserve the unit trace and are positive. The

corresponding states of subsystem S and E are obtained by partial trace over $\mathcal{H}_S$ and $\mathcal{H}_E$, this is, $\rho_S = \mathrm{tr}_E \rho_{SE}$ and $\rho_E = \mathrm{tr}_S \rho_{SE}$. The total Hamiltonian of the closed total system $S + E$ has the general form [4]

$$H(t) = H_S \otimes \mathbb{I}_E + \mathbb{I}_S \otimes H_E + H_I(t), \tag{2.27}$$

where H_S is the Hamiltonian of the open system S, H_E is the Hamiltonian of the environment, and H_I is the Hamiltonian describing the interaction between the system and the environment. A schematic representation is illustrated in Fig. 1.

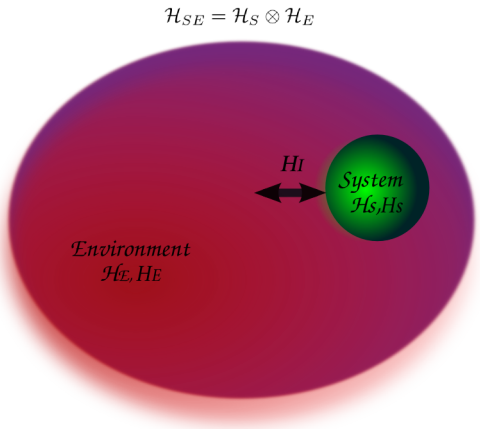

Figure 1 – An open quantum system described by the Hilbert space $\mathcal{H}_S$ and the Hamiltonian H_S, which is coupled to an environment with Hilbert space $\mathcal{H}_E$ and the Hamiltonian H_E through an interaction Hamiltonian H_I.

In the context of the open system theory, the term environment is used for the system coupled to the open system S. The term reservoir refers to an environment with an infinite number of degrees of freedom such that the frequencies of the reservoir modes form a continuum and the term bath or heat bath is used for a reservoir which is in thermal equilibrium state (BREUER; PETRUCCIONE, 2002). The observables referring to S are all of the form $A \otimes \mathbb{I}_E$, where A is an operator acting on the Hilbert space $\mathcal{H}_S$ and $\mathbb{I}_E$ denotes the identity in the Hilbert space $\mathcal{H}_E$. The expectation values of all observables acting on the open system's Hilbert space are

$$\langle A \rangle = \mathrm{tr}_S\{A\rho_S\}. \tag{2.28}$$

The time evolution operator of the total system is given by (BREUER; PETRUCCIONE, 2002):

$$U(t) = \exp[-iHt], \tag{2.29}$$

[4] Here is assumed time independent Hamiltonians. Nevertheless, all concepts discussed here are also valid for time-dependent Hamiltonians.

and the dynamic is obtained from:

$$\rho_{SE}(t) = U(t)\rho_{SE}(0)U^{\dagger}(t). \tag{2.30}$$

Thus, the reduced density operator ρ_S at any time $t \geq 0$ is obtained from the density matrix $\rho(t)$ by mean of

$$\rho_S(t) = \mathrm{tr}_E\{U(t)\rho_{SE}(0)U^{\dagger}(t)\}. \tag{2.31}$$

2.4 Quantum Markov process: Quantum dynamical semigroups

The most important property of a classical, homogeneous Markov process is the semigroup property which is usually formulated in terms of the differential Chapman-Kolmogorov equation (see Appendix A) involving a time-independent generator. The extension of this property to quantum mechanics leads to the concepts of a quantum dynamical semigroup and a quantum Markov process (BREUER; PETRUCCIONE, 2002). In this section, we show this concept.

An important concept in the theory of open quantum systems is that of a dynamical map (BREUER; PETRUCCIONE, 2002). Let us suppose that at the time $t = 0$ the state of the total system $S + E$ is given by an uncorrelated product state

$$\rho_{SE}(0) = \rho_S(0) \otimes \rho_E, \tag{2.32}$$

where $\rho_S(0)$ is the initial state of the reduced system S and ρ_E represents a fixed state of the environment. The expression for the reduced open system state at any time $t \geq 0$ may be written in the form:

$$\rho_S(t) = \mathrm{tr}_E\{U(t,0)[\rho_S(0) \otimes \rho_E]U^{\dagger}(t,0)\}. \tag{2.33}$$

This equation defines a linear map (see Fig. 2.):

$$\Phi(t) : S(\mathcal{H}_{\mathcal{S}}) \to S(\mathcal{H}_{\mathcal{S}}), \tag{2.34}$$

on the open system's state space $\mathcal{S}(\mathcal{H}_{\mathcal{S}})$ which maps any initial open system state $\rho_S(0)$ to the corresponding open system state $\rho_S(t)$ at time t:

$$\rho_S(0) \to \rho_S(t) = \Phi(t)\rho_S(0). \tag{2.35}$$

Here $\Phi(t)$ is called a quantum dynamical map and can be characterized completely in terms of operators pertaining to the open system of the Hilbert space $\mathcal{H}_{\mathcal{S}}$. The dynamical map $\Phi(t)$ maps physical states to physical states, i.e, it preserves the hermiticity and

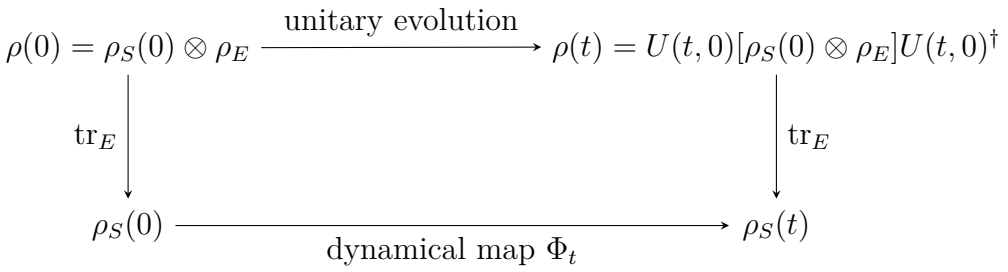

Figure 2 – Schematic construction of dynamical map.

the trace of operators, and it maps positive operators to positive operators (BREUER; PETRUCCIONE, 2002).

Let us assume that $|e_j\rangle$ is an orthonormal basis for the state space of the environment, and let $\rho_E = |e_0\rangle\langle e_0|$ be the initial state of the environment[5]. Thus, Eq. (2.33) can be rewritten as (NIELSEN; CHUANG, 2000)

$$\begin{aligned} \rho_S(t) &= \sum_j \langle e_j|U(t,0)[\rho_S(0) \otimes |e_0\rangle\langle e_0|]U^\dagger(t,0)|e_j\rangle, \\ \rho_S(t) &= \sum_j K_j(t)\rho_S(0)K_j^\dagger(t), \end{aligned} \tag{2.36}$$

where $K_j(t) \equiv \langle e_j|U(t,0)|e_0\rangle$ is an operator on the state space of the principal system. Eq. (2.36) is known as the operator sum representation of $\Phi(t)$. The operators $\{K_j\}$ are known as Kraus operators (KRAUS, 1983). A linear map $\Phi(t)$ that admits a Kraus representation satisfies the next set of axiomatic properties (NIELSEN; CHUANG, 2000; KRAUS, 1983):

- First, $\mathrm{tr}[\Phi(t)\rho] = 1$, for all ρ. The Kraus operators satisfy the completeness relation, this important constraint arises from the requirement that the trace of $\Phi(t)\rho(0)$ be equal to one:

$$\begin{aligned} 1 &= \mathrm{tr}(\Phi(t)\rho(0)) = \mathrm{tr}(\sum_j K_j\rho K_j^\dagger), \\ &= \mathrm{tr}(\sum_j K_j^\dagger K_j\rho). \end{aligned} \tag{2.37}$$

  Since this relationship is true for all ρ, we must have

$$\sum_j K_j^\dagger K_j = \mathbb{I}, \tag{2.38}$$

  for all t. Therefore, the map is a trace-preserving map, since $\Phi(t)$ provides a complete description of the quantum process.

[5] There is no loss of generality in assuming that the environment starts in a pure state.

- Second, $\Phi(t)$ is a convex linear map on the set of density matrices, that is, for probabilities $\{p_i\}$,
$$\Phi(t)(\sum_i p_i\rho_i) = \sum_i p_i\Phi(t)\rho_i. \tag{2.39}$$
- Third, $\Phi(t)$ is a completely positive (CP) map. That is, if $\Phi(t)$ maps density operators of system Q_1 to density operators of system Q_2, then $\Phi(t)A$ must be positive for any positive operator A. Furthermore, if we introduce an extra system R of arbitrary dimensionality, it must be true that $(\mathbb{I}\otimes\Phi(t))(A)$ is a positive for any positive operator A on the combined system RQ_1, where $\mathbb{I}$ denotes the identity map on system R. Complete positivity is a stronger condition than positivity (BREUER et al., 2016). Positivity of a map Φ guarantees that physical states ρ_S are mapped to physical states $\Phi\rho_S$, this means that probabilities stay positive under the action of the map. The stronger property of CP of Φ ensures not only that all physical states of S are mapped to physical states of S, but also that all physical states of $S+R$ are mapped to physical states of $R+S$ [6].

Thus, we conclude that a dynamical map $\Phi(t)$ represents a convex-linear, completely positive and trace-preserving (CPTP) quantum operation. If now the parameter t varies over some time interval from 0 to T, we get a one-parameter family of the dynamical maps (BREUER et al., 2016),

$$\Phi = \{\Phi(t)|0 \leq t \leq T, \Phi(0) = \mathbb{I}\}, \tag{2.40}$$

where $\mathbb{I}$ denotes the unit map, and $\rho_E(0)$ is still keep fixed. This family describes the time evolution of the open system over the time interval $[0, T]$.

The memory effects in the reduced system dynamics can be neglected if the characteristic time scales over which the reservoir correlation functions decay are much smaller than the characteristic time scales of the systematic system evolution. This Markovian behaviour can be formalized with the help of the semigroup[7] property (BREUER; PETRUCCIONE, 2002):

$$\Phi(t_1)\Phi(t_2) = \Phi(t_1+t_2),\ \ t_1, t_2 \geq 0. \tag{2.41}$$

2.5 The Markovian quantum master equation

In this section we describe the theory of quantum master equations, which is a powerful tool complementary to the quantum operators formalism. This approach describes

[6] Here we assume the situation where ρ_S describes the state of some subsystem S of a large system $S+R$

[7] The term semigroup serves to indicate that the family $\{\Phi(t)|0 \leq t \leq T\}$ is, in general, not a full group since the parameter t is restricted to non-negative values.

the quantum noise in continuous time using differential equations. The quantum dynamical semigroup defined in Eq. (2.41) can be presented in exponential form (LINDBLAD, 1976):

$$\Phi(t) = \exp(\mathcal{L}t),\ t \geq 0, \tag{2.42}$$

with a generator $\mathcal{L}$ in Lindblad form. For two arbitrary times, t_1 and t_2, we define the two-parameter family of dynamical maps as:

$$\Phi(t_2, t_1) := \Phi(t_2 - t_1) = \exp[\mathcal{L}(t_2 - t_1)]. \tag{2.43}$$

Considering an intermediate time τ, with $t_1 \leq \tau \leq t_2$, and using the semigroup property given by Eq. (2.41), we obtain:

$$\Phi(t_2, t_1) = \Phi(t_2 - \tau + \tau - t_1) = \Phi(t_2, \tau)\Phi(\tau, t_1). \tag{2.44}$$

Thus, the map $\Phi(t_2, t_1)$ is CP-divisible, i.e., divisible in terms of CP maps for an arbitrary time τ, as consequence here of the semigroup property (BREUER et al., 2016; BREUER; PETRUCCIONE, 2002; BENATTI; CHRUŚCIŃSKI; FILIPPOV, 2017).

The representation Eq. (2.42) leads to the first-order differential equation for the reduced density matrix of the open system,

$$\frac{d}{dt}\rho_S(t) = \mathcal{L}\rho_S(t), \tag{2.45}$$

which is known as the Markovian quantum master equation (BREUER; PETRUCCIONE, 2002). The generator $\mathcal{L}$ of the semigroup represents a super-operator[8]. In order to get the general form for the generator $\mathcal{L}$, it will be necessary to consider a finite-dimensional Hilbert space $\mathcal{H}_S$ ($\dim\mathcal{H}_S = N$). In general terms the spectral decomposition of the density ρ_E of the environment is given by (BREUER; PETRUCCIONE, 2002):

$$\rho_E = \sum_\alpha \lambda_\alpha |\phi_\alpha\rangle\langle\phi_\alpha|, \tag{2.46}$$

where $|\phi_\alpha\rangle$ forms an orthonormal basis in $\mathcal{H}_E$ and the λ_α are non-negative real number with $\sum_\alpha \lambda_\alpha = 1$. Then, using the definition (2.33) we get:

$$\Phi(t)\rho_S = \sum_{\alpha,\beta} W_{\alpha,\beta}(t)\rho_S(0)W^\dagger_{\alpha,\beta}(t), \tag{2.47}$$

where the operators $W_{\alpha,\beta}$ belong to $\mathcal{H}_S$ and are given by:

$$W_{\alpha,\beta}(t) = \sqrt{\lambda_\beta}\langle\phi_\alpha|U(t,0)|\phi_\beta\rangle. \tag{2.48}$$

We continue in this fashion choosing a complete basis of orthogonal operators F_i, with $i = 1, 2, ..., N^2$, that satisfying $\mathrm{tr}_S\{F_i^\dagger F_j\} = \delta_{ij}$. It is convenient to choose traceless

[8] A super-operator is a linear operator acting on a vector space of linear operators.

basis operators (namely $\mathrm{tr}_S F_i = 0$ for $i = 1, 2, ..., N^2 - 1$), except one which is given by $F_{N^2} = (1/N)^{1/2}\mathbb{I}_S$. Applying the completeness relation to each of the operators $W_{\alpha,\beta}(t)$, we obtain (BREUER; PETRUCCIONE, 2002) [9]:

$$W_{\alpha,\beta} = \sum_{i=1}^{N^2} F_i (F_i, W_{\alpha,\beta}). \tag{2.50}$$

Therefore the action of the dynamic map $\Phi(t)$ can be written as:

$$\Phi(t)\rho_S = \sum_{i,j=1}^{N^2} c_{ij}(t) F_i \rho_S F_j^\dagger, \tag{2.51}$$

where

$$c_{ij}(t) \equiv \sum_{\alpha\beta} (F_i, W_{\alpha\beta})(F_i, W_{\alpha\beta})^*. \tag{2.52}$$

Using Eqs. (2.51), the generator $\mathcal{L}$ defined in Eq. (2.42) is given by:

$$\begin{aligned}
\sum_{i=1}^{N^2-1} \mathcal{L}\rho_S &= \lim_{\epsilon\to 0} \frac{1}{\epsilon}\{\Phi(\epsilon)\rho_S - \rho_S\} \\
&= \lim_{\epsilon\to 0}\{\frac{1}{N}\frac{c_{N^2N^2}(\epsilon) - N}{\epsilon}\rho_S + \frac{1}{\sqrt{N}}\sum_{i=1}^{N^2-1}(\frac{c_{iN^2}(\epsilon)}{\epsilon}F_i\rho_S + \frac{c_{N^2i}(\epsilon)}{\epsilon}\rho_S F_i^\dagger) \\
&+ \sum_{i,j=1}^{N^2-1}\frac{c_{ij}(\epsilon)}{\epsilon}F_i\rho_S F_j^\dagger\}.
\end{aligned} \tag{2.53}$$

The coefficients a_{ij} can be defined by:

$$a_{N^2N^2} = \lim_{\epsilon\to 0}\frac{c_{N^2N^2}(\epsilon) - N}{\epsilon}, \tag{2.54}$$

$$a_{iN^2} = \lim_{\epsilon\to 0}\frac{c_{iN^2}(\epsilon)}{\epsilon},\ i = 1, ..., N^2 - 1, \tag{2.55}$$

$$a_{ij} = \lim_{\epsilon\to 0}\frac{c_{ij}(\epsilon)}{\epsilon},\ i, j = 1, ..., N^2 - 1. \tag{2.56}$$

Accordingly, we can write the generator in the form (BREUER; PETRUCCIONE, 2002):

$$\mathcal{L}\rho_S = -i[H, \rho_S] + \{G, \rho_S\} + \sum_{ij=1}^{N^2-1} a_{ij} F_i \rho_S F_j^\dagger. \tag{2.57}$$

[9] The scalar product of the operators A and B is given by:

$$(A, B) \equiv \mathrm{tr}\{A^\dagger B\}. \tag{2.49}$$

where the following quantities have been used:

$$F = \frac{1}{\sqrt{N}} \sum_{i=1}^{N^2-1} a_{iN^2} F_i, \tag{2.58}$$

and

$$G = \frac{1}{2N} a_{N^2N^2} I_S + \frac{1}{2}(F^\dagger + F), \tag{2.59}$$

as well as the Hermitian operator

$$H = \frac{1}{2i}(F^\dagger - F). \tag{2.60}$$

The semigroup is trace-preserving for all density matrices ρ_S. Then,

$$G = -\frac{1}{2} \sum_{ij=1}^{N^2-1} a_{ij} F_j^\dagger F_i. \tag{2.61}$$

It follows that:

$$\mathcal{L}\rho_S = -i[H, \rho_S] + \sum_{ij=1}^{N^2-1} a_{ij}(F_j^\dagger \rho_S F_i - \frac{1}{2}\{F_j^\dagger F_i, \rho_S\}). \tag{2.62}$$

As the coefficient matrix $a = (a_{ij})$ is positive it can be diagonalized through an appropriate unitary transformation u,

$$uau^\dagger = \begin{pmatrix} \gamma_1 & 0 & ... & 0 \\ 0 & \gamma_2 & ... & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & ...\gamma_{N^2-1} & \end{pmatrix},$$

where the eigenvalues γ_i are non-negative. Thus, defining the set of operators A_k by:

$$F_i = \sum_{k=1}^{N^2-1} u_{ki} A_k, \tag{2.63}$$

the following diagonal form of the generator is obtained (BREUER; PETRUCCIONE, 2002):

$$\mathcal{L}\rho_S = -i\left[H_S, \rho_S\right] + \sum_k \gamma_k \left(A_k \rho_S A_k^\dagger - \frac{1}{2} A_k^\dagger A_k \rho_S - \frac{1}{2} \rho_S A_k^\dagger A_k \right). \tag{2.64}$$

This is the most general form for the generator of a quantum dynamical semigroup. The first term represents the unitary dynamics generated by the Hamiltonian H_S. The operators A_k are called Lindblad operators and the corresponding density matrix equation (2.45) is called the Lindblad equation. The non-negative quantities γ_k have the dimension of an inverse time where A_k are taken to be dimensionless. The generator is invariant under the following transformations:

1. Unitary transformations of the set of Lindblad operators,

$$\sqrt{\gamma_i}A_i \to \sqrt{\gamma_i'}A_i' = \sum_i u_{ij}\sqrt{\gamma_j}A_j, \tag{2.65}$$

where u_{ij} is a unitary matrix.

2. Inhomogeneous transformations

$$\begin{aligned} A_i \to A_i' &= A_i + a_i, \\ H \to H' &= H + \frac{1}{2i}\sum_j \gamma_j \left(a_j^* A_j - a_j A_j^\dagger\right) + b. \end{aligned} \tag{2.66}$$

where the a_i are complex numbers and b is real.

Because of the second invariance property it is always possible to choose traceless Lindblad operators.

A time-dependent generalization of Eq. (2.64) can be written in the form (BREUER et al., 2016; BREUER; PETRUCCIONE, 2002):

$$\frac{d}{dt}\rho(t) = \mathscr{K}(t)\rho(t), \tag{2.67}$$

where $\mathscr{K}(t)$ is a time-dependent generator. This generator preserves the Hermiticity and the trace of the density matrix, and can be written as:

$$\begin{aligned} \mathscr{K}(t)\rho_S &= -i\left[H_S(t), \rho_S\right] \\ &+ \sum_k \gamma_k(t)\left(A_k(t)\rho_S A_k^\dagger(t) - \frac{1}{2}A_k^\dagger(t)A_k(t)\rho_S - \frac{1}{2}\rho_S A_k^\dagger(t)A_k(t)\right), \end{aligned} \tag{2.68}$$

where the Hamiltonian $H(t)$, the Lindblad operators $A_k(t)$ and the relaxation rates $\gamma_k(t)$ depend on time. The generator Eq. (2.68) is in the Lindblad form for each fixed $t \geq 0$, if and only if all rates are positive, $\gamma_k(t) \geq 0$ (BREUER et al., 2016; BREUER; PETRUCCIONE, 2002). Thus, processes with $\gamma_k(t) \geq 0$ are defined as time-dependent Markovian even though the dynamical map $\Phi(t_2, t_1)$ does not represent a semigroup, in general (BREUER et al., 2016). A two-parameter family of CPTP maps $\Phi(t_2, t_1)$ can be defined by:

$$\Phi(t_2, t_1) = T_{\leftarrow} \exp[\int_{t_1}^{t_2} dt' \mathscr{K}(t')] \tag{2.69}$$

where $T_{\leftarrow}$ is the chronological time-ordering operator. Time dependent generators no longer lead to a dynamical semigroup. Then, CP-divisibility is not ensured for the dynamical map $\Phi(t_2, t_1)$ in Eq. (2.69). It is achieved, e.g., if $\gamma_k(t) \geq 0$ for all t, since this implies that the generator has the Lindblad form at any fixed time t. However, in general, Eq. (2.69) may violate CP-divisibility.

To summarize: it is important to highlight that the Markovian dynamics is described for dynamical maps satisfying the semigroup condition Eq. (2.41) or, more generally, the CP-divisibility requirement. This may be violated in general for time-local master equations with time-dependent generators. We observe that, recently, other approaches for non-Markovianity in the quantum realm have appeared, such as (POLLOCK et al., 2018). These definitions go beyond the scope of this thesis.

2.6 Quantum noise

In the following subsections we show some examples of open quantum system evolution that can be described in terms of the Kraus representation. They are important in understanding the practical effects of noise in quantum systems.

Depolarizing channel

The depolarizing channel is an important type of quantum noise. It is possible to describe this channel saying that, with probability $1-p$ the qubit is left untouched, while with probability p an "error" occurs. Using the orthonormal basis $\{|0\rangle, |1\rangle\}$ for the qubit, the error can be of any one of three types (PRESKILL, 1998):

- Bit-flip (BF) error: $|0\rangle \to |1\rangle$ & $|1\rangle \to |0\rangle$ or $|\psi\rangle \to \sigma_x|\psi\rangle$ with $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
- Phase-flip (PF) error: $|0\rangle \to |0\rangle$ & $|1\rangle \to -|1\rangle$ or $|\psi\rangle \to \sigma_z|\psi\rangle$ with $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
- Bit-phase-flip (BPF): $|0\rangle \to +i|1\rangle$ & $|1\rangle \to -i|0\rangle$ or $|\psi\rangle \to \sigma_y|\psi\rangle$ with $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

If an error occurs, then ψ evolves to an ensemble of the three states $\sigma_x|\psi\rangle$, $\sigma_y|\psi\rangle$, $\sigma_z|\psi\rangle$, all occurring with equal likelihood. The depolarizing channel can be represented by a unitary operator acting on $\mathcal{H}_{SE} = \mathcal{H}_S \otimes \mathcal{H}_E$ where $\mathcal{H}_E$ is a four dimensional environment. The unitary operator U_{SE} acts according to

$$U_{SE}|\psi\rangle_S|0\rangle_E \to \sqrt{1-p}|\psi\rangle_S|0\rangle_E + \sqrt{\frac{p}{3}}[\sigma_x|\psi\rangle_S|1\rangle_E + \sigma_y|\psi\rangle_S|2\rangle_E + \sigma_z|\psi\rangle_S|3\rangle_E].$$

The environment evolves to one of four mutually orthogonal states that "keep a record" of what transpired (PRESKILL, 1998); We would know the error if we measured the environment in the basis $\{|k\rangle, k = 0, 1, 2, 3\}$ (and we would be able to intervene and reverse

the error). The Kraus representation is given by the evaluation of the trace partial over the environment in the $\{|k\rangle_E\}$. Then,

$$K_k = \langle k|U_{SE}|0\rangle,$$

$$K_0 = \sqrt{1-p}\mathbb{I},\ K_1 = \sqrt{\frac{p}{3}}\sigma_x,\ K_2 = \sqrt{\frac{p}{3}}\sigma_y,\ K_3\sqrt{\frac{p}{3}}\sigma_z. \tag{2.70}$$

A general initial density matrix ρ_S of the qubit evolves according to the equation (NIELSEN; CHUANG, 2000):

$$\rho \to \Phi\rho = \sum_a K_a \rho K_a^\dagger = (1-p)\rho + \frac{p}{3}(\sigma_x\rho\sigma_x + \sigma_y\rho\sigma_y + \sigma_z\rho\sigma_z). \tag{2.71}$$

The effect of the depolarizing channel can be seen in the Bloch representation, where the density matrix for a single qubit is given by Eq.(2.23). If we suppose rotation in $\vec{n} = \vec{z}$, then $\rho = \frac{1}{2}(\mathbb{I} + P_z\sigma_z)$, we have

$$\rho' = \frac{1}{2}[\mathbb{I} + (1 - \frac{4}{3}p)P_z\sigma_z], \tag{2.72}$$

or $P_z' = (1 - \frac{4}{3}p)P_z$. From the rotational symmetry, we see that $P_z' = (1 - \frac{4}{3}p)$ irrespective of the direction in which P points (PRESKILL, 1998). Clearly, the depolarizing process reduces the spin polarization by the factor $(1 - \frac{4}{3}p)$, in this case the Bloch sphere contracts uniformly under the action of the channel, the effect of the depolarizing channel is illustrated in Figure 3a).

Amplitude damping

An important application of quantum operations is the description of energy dissipation effects due to loss of energy from a quantum system, the amplitude damping (AD) noise describes the general behavior of these kinds of processes (NIELSEN; CHUANG, 2000). For the characterization of the AD effect is considered the model of the decay of an excited state of a two-level atom due to spontaneous emission of a photon (PRESKILL, 1998). In this case, the information about the initial state of the atom is obtained by the detection of the emitted photon (observing the environment). The atomic ground state is denoted by $|0\rangle_A$ and the exited state of interest by $|1\rangle_A$. The environment is the electromagnetic field, assumed initially to be in its vacuum state $|0\rangle_E$. There is a probability p that after a time the excited state has decayed to the ground state, and a photon has been emitted, then the environment has made the transition from the no photon state $|0\rangle$ to one photon state $|1\rangle_E$. The unitary transformation that describes this evolution is given by:

$$\begin{aligned} |0\rangle_A \otimes |0\rangle_E &\rightarrow |0\rangle_A \otimes |0\rangle_E, \\ |1\rangle_A \otimes |0\rangle_E &\rightarrow \sqrt{1-p}|1\rangle_A \otimes |0\rangle_E + \sqrt{p}|0\rangle_A \otimes |1\rangle_E, \end{aligned} \tag{2.73}$$

and the Kraus operators are obtained by the evaluation of the partial trace over the environment in the basis $\{|0\rangle_E, |1\rangle_E\}$, which are:

$$K_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix}, \quad K_1 = \begin{bmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{bmatrix}, \tag{2.74}$$

where the operator K_1 induces a "quantum jump", the decay from $|1\rangle_A$ to $|0\rangle_A$, and K_0 describes how the states evolves if no jump occurs. The density matrix evolves as:

$$\begin{aligned} \Phi(\rho) &= K_0 \rho K_0^\dagger + K_1 \rho K_1^\dagger \\ &= \begin{pmatrix} \rho_{00} + p\rho_{11} & \sqrt{1-p}\rho_{01} \\ \sqrt{1-p}\rho_{10} & (1-p)\rho_{11} \end{pmatrix}. \end{aligned} \tag{2.75}$$

Considering that the decay occurs with probability $p = \Gamma\delta t << 1$ in a small-time interval δt, where Γ is the spontaneous decay rate per unit time. The density operator after time $t = n\delta t$ is found applying the channel n times in succession. The ρ_{11} matrix element then decays as $\rho_{11} \rightarrow (1-p)^n \rho_{11}$, the probability that the excited state persists for time t is $(1-\Gamma\delta t)^{t/\delta t} \approx e^{-\Gamma t}$. Hence, we have:

$$\rho(t) = \begin{pmatrix} \rho_{00} + (1-e^{-\Gamma t})\rho_{11} & e^{-\Gamma t/2}\rho_{01} \\ e^{-\Gamma t/2}\rho_{10} & e^{-\Gamma t}\rho_{11} \end{pmatrix}. \tag{2.76}$$

By the time that $t \rightarrow \infty$, the atom is in its ground state with high probability $\rho_{00} \approx 1$. The effect of AD can be visualized in the Bloch representation by the Bloch vector transformation (NIELSEN; CHUANG, 2000)

$$(r_x, r_y, r_z) \rightarrow (r_x\sqrt{1-p}, r_y\sqrt{1-p}, p + r_z(1-p)), \tag{2.77}$$

where every point in the unit ball goes towards the a fixed point at the north pole, where $|0\rangle$ is located. This is shown in the Figure 3b).

In the case of AD noise is assumed that the environment start in the $|0\rangle$ state, this can be seen as the effect of dissipation to an environment at zero temperature. For a quantum process that evolves in a dissipative processes at finite temperature, the process is called generalized amplitude damping (GAD), and it is defined by the follow Krauss operators (NIELSEN; CHUANG, 2000):

$$K_0 = \sqrt{\lambda}\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix}, \quad K_1 = \sqrt{\lambda}\begin{bmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{bmatrix},$$

$$K_2 = \sqrt{1-\lambda}\begin{bmatrix}\sqrt{1-p} & 0\\ 0 & 1\end{bmatrix}, \quad K_3 = \sqrt{1-\lambda}\begin{bmatrix}0 & 0\\ \sqrt{p} & 0\end{bmatrix}, \tag{2.78}$$

where GAD describes the "T_1" relaxation process due to coupling of spins to their surrounding lattice, a large system which is in thermal equilibrium at a temperature often much higher than the spin temperature (NIELSEN; CHUANG, 2000). The effect of GAD in the Bloch representation can be seen through of the Bloch vector transformation (NIELSEN; CHUANG, 2000):

$$(r_x, r_y, r_z) \to (r_x\sqrt{1-p}, r_y\sqrt{1-p}, p(2\gamma-1) + r_z(1-p)). \tag{2.79}$$

Certainly, AD and GAD differ only in the location of the fixed point of the flow, the final state is along the z axis, at the point $(2p-1)$, which is a mixed state (NIELSEN; CHUANG, 2000).

Phase Damping

Phase damping (PD) also called dephasing channel, is a noise process that is uniquely in quantum mechanics, which describes the loss of quantum information without loss of energy (NIELSEN; CHUANG, 2000). In this case, the environment "scatters" off the qubit occasionally with probability p, being kicked into the state $|1\rangle_E$ if A is in the state $|0\rangle_A$ and into the state $|2\rangle_E$ if A is in the state $|1\rangle_A$. Unlike the depolarized channel, this channel picks out a preferred basis for qubit A. This evolution is described by a unitary transformation (PRESKILL, 1998)

$$\begin{aligned} |0\rangle_A &\to \sqrt{1-p}|0\rangle_A \otimes |0\rangle_E + \sqrt{p}|0\rangle_A \otimes |1\rangle_E, \\ |1\rangle_A &\to \sqrt{1-p}|1\rangle_A \otimes |0\rangle_E + \sqrt{p}|1\rangle_A \otimes |2\rangle_E. \end{aligned} \tag{2.80}$$

The Kraus operators are obtained through the evaluation of the partial trace over E in the $\{|0\rangle_E, |1\rangle_E, |2\rangle_E\}$ basis, then:

$$K_0 = (\sqrt{1-p})\mathbb{I}, \; K_1 = \sqrt{p}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \quad K_2 = \sqrt{p}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}. \tag{2.81}$$

One representation in two Kraus operators is possible, then:

$$K_0 = \frac{\sqrt{p}}{2}(\mathbb{I} + \sigma_z), \; K_1 = \frac{\sqrt{p}}{2}(\mathbb{I} - \sigma_z). \tag{2.82}$$

The density matrix evolve as

$$\rho \to \Phi\rho = \sum_a K_a \rho K_a^\dagger = (1 - \frac{1}{2}p)\rho + \frac{1}{2}p\sigma_z\rho\sigma_z. \tag{2.83}$$

In this alternative description, nothing happen to a qubit with probability $(1-\frac{1}{2}p)$ or with probability $\frac{1}{2}p$ the qubit is flipped by the σ_z operation. For the dephasing acting continuously in the time, we are going to suppose that the probability of scattering event per unit time is Γ, so that $p = \Gamma\delta t \ll 1$ for a brief time interval δt. The evolution over a time $t = n\delta t$ is governed by Φ^n. Thus, the off-diagonal terms in the density operator is given by $(1-p)^n = (1-\Gamma t/n)^n \to e^{-\Gamma t}$. Therefore, for initial pure state $\alpha|0\rangle + \beta|1\rangle$. At long times, this state reduce to:

$$\rho(t) = \begin{pmatrix} |\alpha|^2 & e^{-\Gamma t}\alpha\beta^* \\ e^{-\Gamma t}\alpha^*\beta & |\beta|^2 \end{pmatrix} \overset{t\to\infty}{\longrightarrow} \rho(\infty) = \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix}, \tag{2.84}$$

so any phase coherence is lost and the state reduces to a classical, incoherent superposition of populations. The process is called dephasing, because in this process phase coherence is lost, but the energy/population is conserved. The effect of the dephasing channel on the polarization of the qubit is:

$$\rho(\vec{P}) = \frac{1}{2}(\mathbb{I} + \vec{P}.\vec{\sigma}) \to \rho(\vec{P}'), \tag{2.85}$$

where $P'_{1,2} = (1-p)P_{1,2}$, and $P'_3 = P_3$. The Bloch sphere shrinks to a spheroid aligned with the z axis. Under continuous dephasing, the sphere is uniformly contracted in the $x-y$ plane, degenerating to the z axis in the limit of large Γt. The effect of the dephasing channel is showed in Figure 3c).

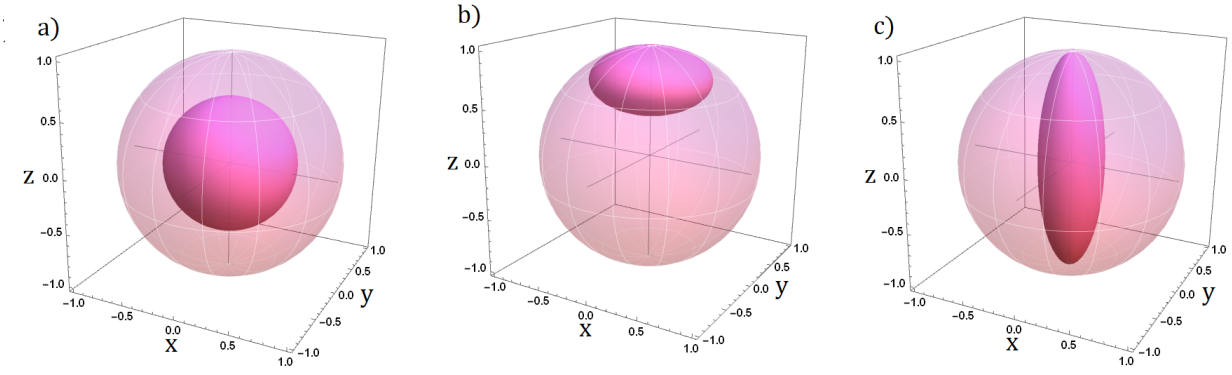

Figure 3 – Qubit under decoherence processes are illustrated with a corresponding shrinking of the Bloch sphere (ligthpurple) for the cases of : a) The effect of the depolarizing channel on the Bloch sphere, for $p = 0.3$. The entire sphere contracts uniformly as a function of p. b) The effect of the AD channel on the Bloch sphere, for $p = 0.7$. The entire sphere shrinks towards the $|0\rangle$ state, c) The effect of the PD channel on the Bloch sphere, for $p = 0.7$. The states on the z axis remain unchanged, while the xy plane is uniformly contracted by a factor $(1-p)$.

3 NON-MARKOVIAN OPEN QUANTUM SYSTEMS

In the previous chapter we discussed the laws that describe the dynamics of open quantum systems derived from the unitary dynamics of the total system. In general, the Markovian behavior is an idealization in the description of the quantum dynamics, where the non-Markovian features has been neglected. The objective of this chapter is to describe some techniques which allow a systematic description of the non-Markovian dynamics of open systems. Conversely to its classical counterpart, there is no universal definition of non-Markovianity in the quantum regime. The classical Markov process defined by the Chapman-Kolmogorov condition (A.10) (see Appendix A) cannot be transferred immediately to the quantum domain because the notion of conditional probability depends on the measurement (BREUER et al., 2016). Therefore, the concept of a quantum non-Markovian process requires a precise definition which cannot be based on classical notions only. The quantitative measure should be at least unitary invariant, experimentally realizable and has a physical interpretation.

In section 3.1, our aim is to provide an exact equation of motion for a non-Markovian open system by projection operator techniques. In section 3.2 we restring our attention to the correlated projection superoperator approaches that are relevant in the scope of this thesis. In section 3.3, we analyze the non-Markovianity measure concept that is independent from the tools to describe their dynamics. In this section we discuss the effect of having initial system-environment correlations in the blackflow of information from the environment into the system. Hence, we focus on one of the most widely studied and significant quantifiers of the degree of non-Markovianity the BLP measure (BREUER; LAINE; PIILO, 2009).

3.1 Nakajima-Zwanzig projection operator method

A general framework to derive exact equations of motion for an open system in a precise mathematical way is provided by projection operator techniques, introduced by Nakajima, Zwanzig and Mori (NAKAJIMA, 1958; ZWANZIG, 1960; MORI, 1965). The main idea underlying the application of projection operator techniques to open quantum systems is to regard the operation of tracing over the environment as a formal projection $\rho \to \mathcal{P}\rho$ in the state space of the total system. A projection superoperator $\mathcal{P}$ is defined to be a linear map $A \to \mathcal{P}A$ where A is an operator acting on $\mathcal{H}$. In order to map any state to valid physical state, $\mathcal{P}$ needs to be at least positive and trace-preserving, this is

$\rho \in \mathcal{S}(\mathcal{H}) \Rightarrow \mathcal{P}\rho \in \mathcal{S}(\mathcal{H})$.

A projection superoperator $\mathcal{P}$ is defined such that $\mathcal{P}\rho$ captures the relevant part of the total density matrix $\rho = \rho_{tot}$ in the interaction picture, which offers an approximate description of the open quantum dynamics. Correspondingly, one defines a projection $\rho \to \mathcal{Q}\rho$ onto the irrelevant part $\mathcal{Q}\rho$, where $\mathcal{Q} + \mathcal{P} = \mathbb{I}$, with $\mathbb{I}$ denoting the unit map. The projection superoperators have the following properties:

$$\mathcal{P}^2 = \mathcal{P} = \mathcal{P}^\dagger, \qquad \mathcal{Q}^2 = \mathcal{Q} = \mathcal{Q}^\dagger, \tag{3.1}$$
$$\mathcal{P}\mathcal{Q} = \mathcal{Q}\mathcal{P} = 0, \qquad \mathcal{P} + \mathcal{Q} = \mathbb{I}.$$

In the context of open quantum systems we also want $\rho_S = \mathrm{tr}_E\{\rho\} = \mathrm{tr}_E\mathcal{P}\rho(t)$. In order to derive a closed equation of motion for the relevant part $\mathcal{P}\rho$, there are basically two different possibilities (BREUER; PETRUCCIONE, 2002). With the methods presented in the following two subsections, one can derive generators for dynamical maps that do not have semigroup (2.41) or CP-divisibility (2.44) properties.

3.1.1 Nakajima-Zwanzig equation

The first, we will study the Nakajima-Zwanzig method (NAKAJIMA, 1958; ZWANZIG, 1960), which leads to an equation for $\mathcal{P}\rho$ that contains a time integration over the past history of the system. Consider the general physical situation of an open system S coupled to an environment E. The dynamics of the combined system $E + S$ is defined by

$$H = H_0 + \alpha H_I, \tag{3.2}$$

where H_0 generates the uncoupled time evolution of the system and environment. The interaction is described by H_I and α stands for a dimensionless expansion parameter. The equation of motion for the density matrix in the interaction representation, is given by [1]

$$\frac{\partial}{\partial t}\rho(t) = -i\alpha[H_I(t), \rho(t)] \equiv \alpha\mathcal{L}(t)\rho(t), \tag{3.3}$$

where the Liouville super-operator is denoted by $\mathcal{L}(t)$. The super-operator $\mathcal{P}$ is defined by:

$$\rho \to \mathcal{P}\rho = \mathrm{tr}_E\{\rho\} \otimes \rho_E \equiv \rho_S \otimes \rho_E, \tag{3.4}$$

where ρ_E is some fixed state of the environment. $\mathcal{P}\rho$ gives the complete information required to reconstruct the reduced density matrix ρ_S of the open system. The super operators $\mathcal{P}$ and $\mathcal{Q}$ are maps in the state space of the combined system $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$. The density matrix ρ_E is a operator in $\mathcal{H}_E$, it may represent a quite arbitrary, but

[1] Here the interaction picture representation of the interaction Hamiltonian is defined by $H_I(t) = \exp(iH_0t)H_I\exp(-iH_0t)$.

known environmental state, called the reference state (BREUER; PETRUCCIONE, 2002). Therefore, the equation for $\mathcal{P}\rho$ is given by:

$$\frac{d}{dt}\mathcal{P}\rho(t) = \int_{t_0}^{t} ds\tilde{K}(t,s)\mathcal{P}\rho(s) + \alpha\mathcal{P}\mathcal{L}(t)\mathcal{G}(t,t_0)\mathcal{Q}\rho(t_0) + \alpha\mathcal{P}\mathcal{L}\mathcal{P}\rho(t). \tag{3.5}$$

This integro-differential equation is called the Nakajima-Zwanzig equation, the detailed derivation of this equation is given in Appendix B.1. Here $\mathcal{L}$ is the Liouvillian corresponding to the von Neumann equation for the total density operator $\rho(t)$ Eq. (3.3). The convolution or memory kernel is defined by

$$\tilde{K}(t,s) = \alpha^2\mathcal{P}\mathcal{L}(t)\mathcal{G}(t,s)\mathcal{Q}\mathcal{L}(s), \tag{3.6}$$

and

$$\mathcal{G}(t,s) = T_{\leftarrow}\exp(\alpha\int_0^t ds'\mathcal{Q}\mathcal{L}(s')), \tag{3.7}$$

where $T_{\leftarrow}$ denotes the chronological time ordering[2]. Also this quantity satisfies the evolution equation

$$\frac{d\mathcal{G}}{dt} = \alpha\mathcal{Q}\mathcal{L}(t)\mathcal{G}(t,s), \text{ with } \mathcal{G}(s,s) = 1. \tag{3.8}$$

The Nakajima-Zwanzig equation (3.5) is an exact equation for the relevant degrees of freedom of the reduced system. The inhomogeneous term $\mathcal{P}\mathcal{L}(t)\mathcal{G}(t,t_0)\mathcal{Q}\rho(t_0)$ depends on the initial condition at time t_0, and this equation involves an integral over the past history of the system in the time interval $[t_0,t]$. Hence, this equation describes non-Markovian memory effects of the reduced dynamics. Remarkably, the Markov limit can be recovered from the Nakajima-Zwanzig equation by some simplifications and approximations. Appendix B.2 contains a brief discussion of this topic.

3.1.2 Time convolutionless projection-operator technique

A second possibility for solving the dynamical equation of $\mathcal{P}\rho(t)$ is the time convolutionless (TCL) projection -operator technique, which departs from Eq. (3.5) to derive an equation that is local in time and has the general form (BREUER; PETRUCCIONE, 2002; BREUER; GEMMER; MICHEL, 2006)

$$\frac{d}{dt}\mathcal{P}\rho(t) = \kappa(t)\mathcal{P}\rho(t) + \mathcal{J}(t)\mathcal{Q}\rho(t_0), \tag{3.9}$$

with the time-local generator, called the TCL generator,

$$\kappa(t) = \alpha\mathcal{P}\mathcal{L}(t)[1-\Sigma(t)]^{-1}, \tag{3.10}$$

[2] It orders any product of super operators such that the time arguments increase from right to left

and the inhomogeneity

$$\mathcal{J}(t) = \alpha \mathcal{P}\mathcal{L}(t)[1 - \Sigma(t)]^{-1}\mathcal{G}(t, t_0)\mathcal{Q}, \tag{3.11}$$

where $\Sigma(t) = \alpha \int_0^t ds \mathcal{G}(t, s)\mathcal{Q}\mathcal{L}_{tot}(s)\mathcal{P}G(t, s)$. The backward propagator of the total system is considered as

$$G(t, s) = T_{\rightarrow} e^{-\alpha \int_s^t ds' \mathcal{L}(s')}, \tag{3.12}$$

with $T_{\rightarrow}$ as the antichronological time-ordering operator. Considering that $\Sigma(t)$ may be expanded into a geometric series:

$$[1 - \Sigma(t)]^{-1} = \sum_{n=1}^{\infty} \Sigma(t)^n, \tag{3.13}$$

and substituting this expression into the kernel Eq. (3.10), it is possible to obtain the kernel equation as a perturbative expansion in α

$$\kappa(t) = \alpha \sum_{n=1}^{\infty} \mathcal{P}\mathcal{L}(t)\Sigma(t)^n\mathcal{P} = \sum_{n=1}^{\infty} \alpha^n \kappa_n(t). \tag{3.14}$$

This expansion can always be assumed, provided that the map is continuous and with a zero initial condition $\Sigma(t_0) = 0$. However, a practical use of such expansion requires that it is truncated at relatively low orders, which may be accurate only at short times and within the weak-coupling regime. After truncation, completely positivity is no longer guaranteed (VEGA; ALONSO, 2017).

3.2 Correlated projection superoperator technique

There are two approaches to choose the projection operator: the standard approach and the correlated projection superoperator technique. Here, we will restrict our attention to the correlated projection superoperator method which lead to develop the non-Markovian framework discussed in the applications sections 5.3 and 6.3. In the standard approach (BREUER; PETRUCCIONE, 2002), the projection superoperator is taken to be $\mathcal{P}\rho = \rho_S(t) \otimes \rho_0$, where $\rho_S(t) = \mathrm{tr}_E\{\rho(t)\}$ represents the reduced density matrix of the open system, tr_E denoting the trace over the environment and ρ_0 is some fixed environmental state. This superoperator is suitable for those problems in which system-environment correlations are small both initially and during the evolution, so they can be treated as small perturbations of the reduced density matrix.

On the other hand, the correlated projection superoperator technique (BREUER; GEMMER; MICHEL, 2006; BREUER, 2007) considers the relevant part of the dynamics as a correlated system-environment state. This second option is naturally adapted to those situations in which system and environment states are non-negligibly correlated initially

and/or during the dynamics. The relevant part of the dynamics is expressed in terms of a positively correlated projection superoperators

$$\mathcal{P} = \mathbb{I}_S \otimes \Lambda, \tag{3.15}$$

where Λ is a linear map that takes operators on $\mathcal{H}_E$ to operators in $\mathcal{H}_E$. A projection superoperator of this form leaves the system S unchanged and acts nontrivially only on the variables of the environment E. Given that $\mathcal{P}$ is positive, the map Λ[3] must be N_S-positive, where N_S is the dimension of $\mathcal{H}_S$. The map Λ can be represented in terms of environment Hermitian operators A_i and B_i, satisfying $\mathrm{tr}_B\{A_j B_i\} = \delta_{ij}$ (BREUER, 2007). These operators should fulfill certain properties so that Λ is a CPTP map. In this representation

$$\mathcal{P}\rho(t) = \sum_i \mathrm{tr}_E\{A_i \rho(t)\} \otimes B_i. \tag{3.16}$$

An example of a projection superoperator is obtained with the choice

$$A_i = \Pi_i \qquad \text{and} \qquad B_i = \frac{\Pi_i \rho_0 \Pi_i}{Z_i},$$

where $i = 1.....n$(n being the total number of operators in the expansion), $Z_i = \mathrm{tr}_E\{\Pi_i \rho\}$, and Π_i are the projection operators on $\mathcal{H}_\mathcal{E}$ such that $\Pi_i \Pi_j = \delta_{ij}\Pi_i$, and $\sum_i \Pi_i = \mathbb{I}_E$. The explicit form of the projection superoperator is given by

$$\mathcal{P}\rho = \sum_i \mathrm{tr}_E\{\Pi_i \rho\} \otimes \frac{\Pi_i \rho_0 \Pi_i}{Z_i}, \tag{3.17}$$

and the reduced density matrix is described as a sum of a set of unnormalized states $\rho_i(t)$:

$$\rho_S(t) = \mathrm{tr}_E\{\mathcal{P}\rho(t)\} = \sum_i \rho_i(t), \tag{3.18}$$

where $\mathrm{tr}_S \rho_S = 1$. The states $\rho_i(t) = \mathrm{tr}\{\Pi_i \rho(t)\}$ belong to a subspace of the total space $\mathcal{H}$ and reflect correlations between the system and the environment.

Considering an initial condition of the form $\rho(0) = \mathcal{P}\rho_0 = \sum_i \rho_i(0) \otimes B_i$ and using the TCL technique, a system of equations for each ρ_i is obtained, each with the general form

$$\frac{d}{dt}\rho_i = \kappa_i(t)(\rho_1, ..., \rho_n), \tag{3.19}$$

where the time-dependent generators $\kappa_i(t)$ can be approximated as a time-independent one κ_i following a Markov approximation. Note that while in the standard approach this is linked to Born approximation, implying zero system-environment correlations at the second order in the system-environment coupling, this is not the case in the present

[3] Λ is a projection $\Lambda^2 = \Lambda$, must be CPTP map.

derivation. After this approximation, a generalized Lindblad equation can be obtained (BUDINI, 2006a; BREUER, 2007)

$$\frac{d}{dt}\rho_i = -i[H_i, \rho_i] + \sum_{j\lambda}(R_\lambda^{ij}\rho_j R_\lambda^{ij\dagger} - \frac{1}{2}\{R_\lambda^{ij\dagger}R_\lambda^{ij}, \rho_i\}). \tag{3.20}$$

Here H_i and R_λ^{ij} are system Hermitian operators. The general structure of a master equation (3.20) which governs the dynamics of the ρ_i, models strong non-Markovian effects, while the physical conditions of normalization and positivity are preserving (BREUER, 2007).

Let us now consider the proposal of Budini (2006a) . In this framework it is considered a projection of the form Eq. (3.17), with ρ_0 being the stationary state of the bath. The notation Π_R is used to refer to the projections to each subspace (hence $i \equiv R$). The projectors $\Pi_R = \sum_{\epsilon_R} |\epsilon_R\rangle\langle\epsilon_R|$ decompose the Hilbert space of the environment into different subreservoirs, each spanned by the base of eigenvectors $|\epsilon_R\rangle$. In other words, this projector $\mathcal{P}$ takes into account that each bath-subspace associated to the projectors Π_R induces a different system dynamics, each of which represented by the states $\rho_R(t)$. Each subspace can be seen as a subreservoir. Hence, this projection corresponds to splitting the environment into a set of subreservoirs, such that the interaction Hamiltonian can be written as a direct sum of Hamiltonians:

$$H_I = \textstyle\sum_{R,R'} H_{I_{R,R'}} \text{ with } H_{I_{R,R'}} = \Pi_R H_I \Pi_{R'}.$$

This choice gives rise, in the long time limit, to the same general Eq. (3.20), which connects each ρ_R to the other ρ'_R $(R' \neq R)$ (VEGA; ALONSO, 2017). Note that when $H_{I_{R,R'}} = 0$ for $(R' \neq R)$, the interaction Hamiltonian can be written as a direct sum of sub-Hamiltonians for each subspace

$$H_I = H_{I_1} \bigoplus H_{I_2} ..... H_{I_R} \bigoplus H_{I_{R+1}} ... \tag{3.21}$$

and writing the evolution of the total density matrix $\rho(t)$ in the interaction representation and splitting the full dynamics in contributions $\mathcal{P}\rho(t)$ and $\mathcal{Q}\rho(t)$, up to second order in the interaction Hamiltonian we obtain (see Eq. (B.13) in Appendix B):

$$\frac{\partial}{\partial t}\mathcal{P}\rho(t) = \int_{t_0}^{t} ds \mathcal{P}\mathcal{L}(t)\mathcal{L}(s)\mathcal{P}\rho(s), \tag{3.22}$$

where $\mathcal{L}(t)$ is the total Lioville operator in a interaction representation. Here the initial state is assumed uncorrelated, $\rho(0) = \rho_S(0) \otimes \rho_E$. From this equation and Eq. (3.21), each state $\rho_R(t)$ in a Schrödinguer representation, evolves as (BUDINI, 2006a; BUDINI, 2008):

$$\frac{d\rho_R(t)}{dt} = -i[H_S, \rho_R(t)] - \int_0^\infty ds \text{tr}_{E_R}\{[H_{I_R}, [H_{I_R}(-s), \rho_R(t) \otimes \rho_{E_R}]]\} \tag{3.23}$$

with $\rho_{E_R} = \frac{\Pi_R \rho_E \Pi_R}{P_R}$. This evolution without taking into account the transients of order of the correlation time of the sub-bath, can be approximated by a Lindblad equation:

$$\frac{d\rho_R(t)}{dt} = \mathcal{L}_H \rho_R(t) + \gamma_R \mathcal{L} \rho_R(t), \tag{3.24}$$

where $\mathcal{L}_S[\bullet] = -i[H_S, \bullet]$ and the dissipative contribution is given by the Lindbland superoperator

$$\mathcal{L}[\bullet] = \frac{1}{2} \sum_{\alpha,\beta} a_{\alpha,\beta}([V_\alpha, \bullet V_\beta^\dagger] + [V_\alpha \bullet, V_\beta^\dagger]. \tag{3.25}$$

Thus, each $\rho_R(t)$ follows a Lindblad type of evolution induced by the coupling with the corresponding subreservoir, and independently of other $\rho_{R'}(R' \neq R)$. The reduced density operator of the system is obtained by (BUDINI, 2008):

$$\rho_S(t) = \mathrm{tr}_E[\mathcal{P}\rho(t)] = \sum_R P_R \rho_R(t), \tag{3.26}$$

where the statistical weights satisfy $\sum_R P_R = 1$ and are given by :

$$P_R = \mathrm{tr}_E[\rho_{RE}] = \sum_{\epsilon_R} \langle \epsilon_R | \rho_E | \epsilon_R \rangle. \tag{3.27}$$

It is worth noting that, the fact that ρ_R follows a Markovian evolution does not mean that ρ_S will also do so. Indeed, the evolution of ρ_S has the form of a convoluted master equation as long as the weights P_R are different. The average of ρ_S over the set $\{\gamma_R, P_R\}$ can be written as:

$$\frac{d\rho_S(t)}{dt} = \mathcal{L}_S \rho_S(t) + \int_0^t d\tau \mathbb{L}(t-\tau)\rho_S(\tau), \tag{3.28}$$

with the superoperator $\mathbb{L}$ satisfying

$$\langle G_R(u)\gamma_R \mathcal{L} \rangle [\bullet] = \langle G_R(u) \rangle \mathbb{L}(u)[\bullet] \tag{3.29}$$

Here, $G_R(u)$ is the Markovian propagator of each state $\rho_R(t)$, given by $G_R(u) \equiv [u - (\mathcal{L}_S + \gamma_R \mathcal{L}]^{-1}$. Eq. (3.28) may represent a variety of system decay behaviors and also many structures of non-local Lindbland equations. The superoperator $\mathbb{L}(t)$ simplified in a effective approximation is given by:

$$\mathbb{L}(u) \simeq \tilde{K}(u - \mathcal{L}_S)\mathcal{L}, \tag{3.30}$$

which corresponds to the approximate solution of Eq. (3.29) discarding the dependence introduced by $\mathcal{L}$ in the propagator $G_R(u)$, namely, $\mathcal{L}_R \to -\mathbb{I}$. It follows that in the effective approximation Eq. (3.28) can be written as (BUDINI, 2005; BUDINI, 2008):

$$\frac{d}{dt}\rho_S(t) \approx \mathcal{L}_S(\rho_S) + \int_0^t ds \tilde{K}(t-\tau) e^{(t-\tau)\mathcal{L}_S} \mathcal{L}\rho_S(\tau). \tag{3.31}$$

Here $\tilde{K}(t-\tau)$ is a superoperator that depends on the rates γ_R and the probabilities P_R. The formalism showed here will be used in the specific applications in the remaining Chapters.

3.3 Non-Markovianity measures: BLP measure

In general terms, a measure of non-Markovianity is a function which assigns a number (nonnegative) to each dynamic, in such a way that the zero value is obtained if and only if the dynamics is Markovian. In this section let us consider the measure of non-Markovianity constructed by Breuer, Laine and Piilo (BREUER; LAINE; PIILO, 2009), which is based on the study of the dynamical behavior of the distinguishability of states. The essential idea is that distinguishability between two states reduces continuously in a Markovian process, whereas in non-Markovian dynamics one essential property is the growth of this distinguishability. This measure of non-Markovianity is based in the trace distance of two quantum states, which describes the probability of successfully distinguishing these states.

A trace distance is a measure for the distance between two quantum states ρ_1 and ρ_2 given by (NIELSEN; CHUANG, 2000):

$$D(\rho_1, \rho_2) = \frac{1}{2} \parallel \rho_1 - \rho_2 \parallel, \tag{3.32}$$

where the trace-norm of the trace class operator A is defined by $\parallel A \parallel = \mathrm{tr}|A|$ and the modulus of the operator is given by $|A| = \sqrt{A^\dagger A}$. The trace distance D represents a metric on the space of density matrices, satisfying $0 \leq D \leq 1$, where $D = 0$ if and only if $\rho_1 = \rho_2$, and $D = 1$ if and only if ρ_1 and ρ_2 are orthogonal[4]. It is clearly a symmetric function of the two states $D(\rho_1, \rho_2) = D(\rho_2, \rho_1)$. Furthermore, the trace distance is preserved under unitary transformations, $D(U\rho_1 U^\dagger, U\rho_2 U^\dagger) = D(\rho_1 \rho_2)$. The trace distance plays an important role in quantum information theory (NIELSEN; CHUANG, 2000), this measure is a useful tool that will be explored in the next chapters of this thesis.

The first property of the trace distance is related with the distinguishability. Suppose that Alice prepares a quantum system in one of two states ρ_1 and ρ_2, each with probability $\frac{1}{2}$, and sends the system to Bob who performs a measurement to decide whether the system was in the state ρ_1 or ρ_2. The probability that Bob can successfully identify the state of the system is given by (NIELSEN; CHUANG, 2000):

$$P = \tfrac{1}{2}[1 + D(\rho_1, \rho_2)],$$

where the trace distance represents the bias in favor of the correct state discrimination by Bob. Thus, the trace distance can be interpreted as a measure of the distinguishability of two quantum states. For example, if Alice prepared two orthogonal states, we have $D = 1$, in this case we obtain $P = 1$, which is well known that orthogonal states can be distinguished with certainty by a single measurement.

[4] Two density matrix are said to be orthogonal if their supports (the subspaces spanned by their eigenstates with nonzero eigenvalue) are orthogonal.

The second remarkable property of the trace distance is given by the fact of that all CPTP maps Φ are contractible for this distance (BREUER et al., 2016; BREUER; LAINE; PIILO, 2009; RUSKAI, 1994):

$$D(\Phi\rho_1, \Phi\rho_2) \leq D(\rho_1, \rho_2). \tag{3.33}$$

From this inequality we can conclude that a trace-preserving quantum operation can never increase the distinguishability of any two quantum states. The equality in Eq. (3.33) holds if Φ is a unitary transformation.

Let us now suppose that Alice prepares her states $\rho_{1,2}(0)$ as initial states of an open quantum system S coupled to some environment E. Bob will then receive at time t the system in one of the states $\rho_{1,2}(t) = \Phi(t)\rho_{1,2}(0)$, where $\Phi(t)$ denotes the corresponding CPTP quantum dynamical map. Using the semi-group property $\Phi(\tau + t) = \Phi(\tau)\Phi(t)$, we obtain from the Eq. (3.33) that for all $t \geq 0$,

$$D(\rho_1(t), \rho_2(t)) \leq D(\rho_1(0), \rho_2(0)). \tag{3.34}$$

Thus, for all CPTP maps $\Phi(t)$ the trace distance is a monotonically decreasing function of the time. This behavior can be interpreted like a loss of information from the open system into the environment, this is a general feature of quantum Markov processes, implying that under a Markovian evolution any two states generally become less and less distinguishable as time increases. This interpretation leads to the following definition (BREUER et al., 2016): A quantum process given by a family of quantum dynamical maps $\Phi(t)$ is said to be Markovian if the trace distance $D(\rho_1(t), \rho_2(t))$ corresponding to all pair of initial states $\rho_1(0)$ and $\rho_2(0)$ decreases monotonically for all times $t \geq 0$. In a quantum Markovian process the open system continuously loses information to the environment.

On the other hand, a quantum process is non-Markovian if there are a pair of states $\rho_1(0)$ and $\rho_2(0)$ such that the trace distance $D(\rho_1, \rho_2)$ is nonmonotonic, that is, starts to increase for some time $t > 0$. A non-Markovian process is characterized by a flow of information from the environment back into the open system, which clearly express the presence of memory effects: information contained in the open system is temporarily stored in the environment and comes back at a later time to influence the system (BREUER et al., 2016; BREUER; LAINE; PIILO, 2009). According to this definition Markovianity and non-Markovianity is a property of the dynamical maps Φ describing the open system dynamics. Using this open quantum system scenario, we associated the dynamical non-Markovianity to reservoir memory effects. The measure for the degree of memory effects is defined by (BREUER; LAINE; PIILO, 2009):

$$\mathcal{N}(\Phi) = \max_{\rho_{1,2}(0)} \int_{\sigma>0} dt\sigma(t, \rho_{1,2}(0)), \tag{3.35}$$

where

$$\sigma(t, \rho_{1,2}(0)) = \frac{d}{dt} D(\rho_1(t), \rho_2(t)), \tag{3.36}$$

denotes the rate of change of the trace distance of the evolved pair of states. Hence, a process is said to be non-Markovian if there exist a pair of initial states $\rho_{1,2}(0)$ and a certain time t such that $\sigma(t, \rho_{1,2}(0)) > 0$. In Eq. (3.35) the integral is extended to all time intervals over which $\sigma(t, \rho_{1,2}(0)) > 0$ and the maximum is taken over all pairs of initial states of the open system's state space $\mathcal{S}(\mathcal{H}_\mathcal{S})$. We have $\mathcal{N}(\Phi) = 0$ if and only if the process is Markovian.

In this section we showed one of the quantifiers used in this thesis for measuring the degree of non-Markovianity, i.e, the BLP measure. In order to define the other measure used in this thesis, the RHP measure (RIVAS; HUELGA; PLENIO, 2010), we need to introduce the concepts of the quantum correlations. Thus, in the next chapter after the discussion about quantum correlations, the RHP measure will be formally defined and the concepts of non-Markovianity showed in this Chapter will be expanded in the quantum correlation context.

4 QUANTUM CORRELATIONS

In the early days of quantum information, the entanglement was considered as the qualitative feature of quantum theory that most strikingly distinguishes it from the classical realm (PLENIO; VIRMANI, 2005). The development of Bell's inequalities has made this distinction quantitative, leading the non-local features of quantum theory to become accessible to experimental verification. Bell's inequalities may indeed be viewed as an early attempt to quantify the quantum correlations (PLENIO; VIRMANI, 2005). Werner in (1989) proved that there are entangled quantum states that generate outcomes in perfect agreement with a local-realistic view. Zurek (2000), Henderson and Vedral (2001) and Olliver and Surek (2001) concluded that the entanglement does not account for all non-classical correlations and that even separable states usually contain correlations that are not entirely classical. These correlations are called quantum discord. Our intention in this chapter is to outline the concept of quantum correlation. Quantum systems can be correlated in ways inaccessible to classical objects and the existence of non-classical correlations in a system can be seen as a signature that subsystems are genuinely quantum (MODI et al., 2012). In another way, quantum correlations are also responsible for decoherence and dissipation of quantum systems (BREUER; PETRUCCIONE, 2002).

This chapter is organized as follows: In section 4.1 we briefly sketch the concept of entanglement. In section 4.2 we review definitions of quantum correlation and some general properties. In Section 4.3 we discuss the quantification of quantum correlation present in any quantum state based in the geometric quantum discord (GQD). Some results are given for Bell diagonal states. Finally, in section 4.4 we review the several criteria for generalized measures of quantum correlations.

4.1 Entanglement

In the context of quantum information the classical correlations are defined via local operations and classical communications (LOCC). Therefore, for a quantum system that exhibits correlations that cannot be simulated classically, it is usual to attribute them to quantum effects, and hence these correlations are called quantum correlations (PLENIO; VIRMANI, 2005). If a mixed state is prepared by all the parties in a "classical" way, that is, can trivially be created by LOCC, this state is said to be separable (HORODECKI et al., 2009). Mathematically, a state represented by the density matrix ρ is said to be separable, if it can be written in the form (PLENIO; VIRMANI, 2005):

$$\rho_{ABC\ldots} = \sum_i p_i \rho_A^i \otimes \rho_B^i \otimes \rho_C^i \otimes \ldots \tag{4.1}$$

where p_i is a probability distribution. In practice, it is hard to decide if a given states is separable or entangled basing on the definition itself (HORODECKI et al., 2009). Here, it is not our purpose to discuss the separability problem. Entangled states can be defined also like the ones that cannot be simulated by classical correlations (MASANES; LIANG; DOHERTY, 2008). This interpretation defines entanglement in terms of the behavior of the states rather than in terms of preparation of the states. Among the main properties of entanglement, we have that LOCC cannot create entanglement from an unentangled state and that the entanglement of states does not increase under LOCC transformations (PLENIO; VIRMANI, 2005). From the previous propriety follows that entanglement does not change under local unitary operations. The maximally entangled state exists at least in two-party systems consisting of two fixed d-dimensional subsystems. These states are given by (PLENIO; VIRMANI, 2005):

$$|\phi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d-1} |i\rangle|i\rangle. \tag{4.2}$$

The Bell states are specific cases of bipartite maximally entangled states. For bipartite systems in the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, the four Bell states defined as (HORODECKI et al., 2009)

$$|\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle \pm |1\rangle|0\rangle), \quad |\phi^\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle \pm |1\rangle|1\rangle). \tag{4.3}$$

Thus, if one measures only at one of the subsystems of these states one finds it with equal probability in state $|0\rangle$ or the state $|1\rangle$. However, the result of the measurements for both subsystems are perfectly correlated.

In the above discussion, the set of classical states is exactly the set of separable quantum states and quantum correlations correspond exactly to entanglement. However, this notion of classicality can be argued considering the nature of the operations allowed in the framework of LOCC (MODI et al., 2012). The aim of the following section is to show certain notions, according to which the classical states form a subset of the separable states.

4.2 Quantum discord

Two systems are correlated if together they contain more information than taken separately. If we measure the lack of information than taken separately. The lack of information can be measured by the entropy. In classical information theory (COVER; THOMAS, 2006) the entropy $H(X)$, known as Shannon entropy, describes the ignorance about a classical random variable X. If X takes the value x with probability p_x, the

Shannon entropy is then given by:

$$H(X) = -\sum_x p_x \log p_x. \tag{4.4}$$

Therefore, if we consider two classical random variables X and Y, for which the joint probability distribution of getting outcome $X = x$ and $Y = y$ is $p_{x,y}$, the correlation between X and Y is measured by the mutual information:

$$I(X : Y) \equiv H(X) + H(Y) - H(X, Y). \tag{4.5}$$

For classical variables, Baye's rule defines a conditional probability as $p_{X|Y=y} = p_{X,Y=y}/p_{Y=y}$. Consequently, the classical mutual information can be written in an equivalent form:

$$J(X|Y) = H(X) - H(X|Y), \tag{4.6}$$

where the conditional entropy $H(X|Y) = \sum_y p_y H(X|Y = y)$ is the average of entropies $H(X|Y = y) = -\sum_y p_{X|Y=y} \log p_{X|Y=y}$. The classical correlation can therefore be interpreted as information gained about one subsystem as a result of a measurement on the other. Fig. 4 depicts this relationship in a graphical manner.

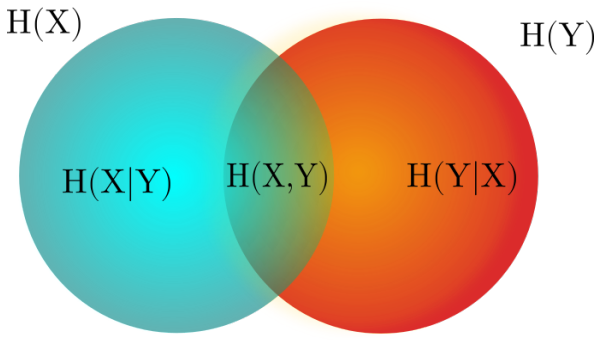

Figure 4 – Conditional entropy. The Venn diagram shows the join entropy $H(X, Y)$, marginal entropies $H(X)$ and $H(Y)$ and conditional entropies $H(X|Y)$ and $H(Y|X)$ for a joint classical probability distribution for (correlated) random variables X and Y.

These definitions of classical mutual information can be analyzed in the quantum domain (ZUREK, 2003; OLLIVIER; ZUREK, 2001; HENDERSON; VEDRAL, 2001). The quantum mutual information can be defined by replacing the classical probability distributions by the appropriate density matrices $\rho_A, \rho_B, \rho_{AB}$ and the Shannon entropy by

von Neumann entropy [1] given by:

$$S(A) = S(\rho_A) = -\mathrm{tr}(\rho_A \log_2 \rho_A), \tag{4.7}$$

in the classical definition Eq. (4.5). In this order, we obtain:

$$I_{AB} \equiv S(A) + S(B) - S(AB), \tag{4.8}$$

where $S(A) + S(B)$ represents the uncertainty of ρ_A and ρ_B treated separately[2], and $S(AB)$ is the uncertainty about of combined system given by ρ_{AB}. The generalization of J is not automatic as for I due the conditional entropy $H(A|B)$. This quantity measures the uncertainty about A after we measure B. Depending on the observable we choose to measure B, the value of the conditional entropy would be different as well as negative. Conversely to the classical case, in the quantum analogue there are many different measurements that can be performed on a system and measurements generally disturb the quantum state. Hence, assuming that measurements on subsystem B are defined by a set of local projective measurements $\{\Pi_j^B\} = \{|j_B\rangle\langle j_B|\}$ where the label j distinguishes different classical outcomes of this measurement, the state of A related to the measurement Π_j^B is

$$\rho_{A|\Pi_j^B} = \frac{(I_A \otimes \Pi_j^B)\rho_{AB}(I_A \otimes \Pi_j^B)}{p_j}, \tag{4.9}$$

where $p_j = \mathrm{tr}_{AB}((I_A \otimes \Pi_j^B)\rho_{AB}(I_A \otimes \Pi_j^B))$ is the probability of obtaining the outcome j and I_A is the identity matrix of the subsystem A. The entropies $H(\rho_{A|\Pi_j^B})$, weighted by probabilities p_j, provide to the conditional entropy of A given the complete measurement $\{\Pi_j^B\}$ on B:

$$H(A|\{\Pi_j^B\}) = \sum_j p_j H(\rho_{A|\Pi_j^B}). \tag{4.10}$$

This allows, in analogy with Eq. (4.6), the quantum generalization of the classical correlation of the state ρ_{AB} (HENDERSON; VEDRAL, 2001):

$$J(A|B) \equiv H(A) - H(A|\{\Pi_j^B\}). \tag{4.11}$$

This quantity represents the information gained about the system A as a result of the measurement $\{\Pi_j^B\}$. To quantify the classical correlations of the state independently of a

[1] The von Neumann entropy of a density matrix ρ is given by $S(\rho) = -\mathrm{tr}(\rho \log_2 \rho)$, which reduces to $-\sum_i \lambda_i \log_2 \lambda_i$, where λ_i are the eigenvalues of ρ.

[2] Here $\rho_i = \mathrm{tr}_j(\rho_{AB})(\{i,j\} \in \{A,B\}, i \neq j)$ are local density matrices of ρ_{AB}.

measurement $J(A|\{\Pi_j^B\})$ is maximized over all measurements[3],

$$J(A|B) \equiv \max_{\{\Pi_j^B\}} J(A|\{\Pi_j^B\}). \tag{4.12}$$

The quantum discord (OLLIVIER; ZUREK, 2001) of a state ρ_{AB} under a measurement $\{\Pi_j^B\}$ is defined as a difference between total correlations (which are identical in the classic case), as given by the quantum mutual information in Eq. (4.5) and the classical correlations Eq. (4.12):

$$\begin{aligned} D(B|A) &\equiv I(A:B) - J(B|A) && (4.13)\\ &= \min_{\{\Pi_j^B\}} \sum_j p_j H(\rho_{A|\{\Pi_j^B\}}) + S(B) - S(AB). && (4.14) \end{aligned}$$

The minimization here is equivalent to maximization in Eq. (4.12). It is possible to say that Eq. (4.13) reveals the quantumness of the correlations between the partitions A and B since this shows the departure between the quantum and the classical versions of information theory. Remarkably, quantum discords captures the non-classical aspects of correlations contained in certain states, which include entanglement. However, while an entanglement measure vanish for separable states, quantum discord can be non-zero for these states.

Let us briefly enumerate some properties of the quantum discord (MODI et al., 2012):

1. It is not symmetric. In general $D(B|A) \neq D(A|B)$ [4]. This is expected because conditional entropy is not symmetric.
2. Discord is nonnegative $D \geq 0$, since $I_{AB} \geq J_{A|B}$.
3. Discord is invariant under local unitary transformations, that is, it is the same for the state ρ_{AB} and state $(U_A \times U_B)\rho_{AB}(U_A \times U_B)^\dagger$, for arbitrary unitaries U_A and U_B on the subsystems A and B.
4. Discord $D(B|A)$ vanish if and only if the state is classical-quantum.
5. For a bipartite pure state, quantum discord reduces to entanglement. As given by the von Neumann entropy.
6. Discord is bounded from above by the von Neumann entropy of the measure subsystem B i.e. $D(\rho_{AB}) \leq S(\rho_B)$ while $J(A|B) \leq \min\{S(\rho_A), S(\rho_B)\}$.

[3] For the Henderson-Vedral classical correlation the maximum is taken over the complete set of positive operator value measurements (POVM) $\{\Pi_j^B\}$ on subsystem B.

[4] Sometimes is used the notation $D^{\leftarrow}(\rho_{AB})$, where the superscript denotes that the measurement has been performed in the subsystem B while $D^{\rightarrow}$ denotes quantum discord for the measurement in the first subsystem A.

4.2.1 Classical States

In this section, we are going to discuss some relevant aspects about the set of classical states. States that have zero discord with respect to one or more parties, known as classical states, are important for several reasons (MODI et al., 2012):

- Vanishing discord corresponds to a key notion of classicality, for which maximal information about a subsystem can be obtained by some specific local measurement without altering correlations with the rest of the system.
- Zero-discord states have application to the theory of decoherence where they describe the classical correlations between the pointer states of some measurement apparatus and the internal quantum states, which results from interaction with the environment. (We will describe the pointer states in the section 5.3.2).
- Discord measures can be defined using the set of classical states, as is the case for the relative entropy of discord and the geometric quantum discord.

The next theorem characterizes the zero-discord states (MODI et al., 2012; DATTA, 2010):

Theorem 1. *A state ρ_{AB} satisfies $D(A|B) = 0$ if and only if there exists a complete set of rank-one orthogonal projectors Π_a on A, satisfying $\sum_a \Pi_a = \mathbb{I}$ and $\Pi_a \Pi_{a'} = \delta_{aa'}\Pi_a$, such that:*

$$\rho_{AB} = \sum_a p_a \Pi_a^A \otimes \rho_a^B. \tag{4.15}$$

Which is a linear combination of the tensor products of the orthogonal projector Π_a^A in $\mathcal{H}_A$ and an arbitrary density operator ρ_a^B in $\mathcal{H}_B$, with $0 \leq p_a \leq 1$ ($\sum_a p_a = 1$).

The set of states classical with respect to A is denoted as $\mathcal{C}_A$. A physical interpretation of Eq. (4.15) is that if we have any state in $\mathcal{C}_A$ there exist a basis for A for which the locally-accessible information is maximal and, from the perspective of an external observer, this information can be obtained without disturbance to the combined system. Remarkably, if in a given theory all separable states have null discord, then all pure states are perfectly distinguishable. As a consequence it is possible to conclude that discord is not only a signature of "quantumness", rather is a signature of nonclassicality.

4.3 Geometric quantum correlation measures

Quantification of quantum correlation present in any quantum state is one of the fundamental tasks related to the understanding and efficient utilization of the state for diverse quantum information processing schemes. The concept of the quantum discord

presented in the last section is based in the quantizing of certain concepts of classical information theory. This definition involves optimization over sets of local measurements, where the analytical computation is in general a difficult task.

In the last years have been proposed disord-like quantum correlation measures that can be categorized roughly into two different families, this is, those based on entropy theory (MODI et al., 2012) and those based on various distance measures of quantum states (BENGTSSON; ZYCZKOWSKI, 2006). In our thesis, we focus in the geometric quantum discord (GQD), which belong to the distance-based formulation of quantum discord. The minimization involved in this definition can often be performed explicitly and in this way it becomes a convenient tool for analysing quantum correlation associated with the system. The main idea here is to consider a set of states that are devoid of quantum correlations in some sense. Quantum correlation of a given state is then defined as the minimal distance of the state from that set (BERA et al., 2017).

In this order, the category of geometric quantum discord of a state can be characterized by its closest (pseudo) distance to the zero-discord state in set $\mathcal{C}_A$ (see subsection 4.2.1). Likewise, it is possible to use the quantum-classical states and define the GQD with respect to party B, or the classical-classical states and define the GQD with respect to AB. Hence, a general form of the GQD can be written as:

$$D_{\mathcal{D}(\rho)} = \min_{\rho_c \in \mathcal{C}_A} \mathcal{D}(\rho, \rho_c), \tag{4.16}$$

where $\mathcal{D}(\rho, \rho_c)$ is the proper distance measure of states, e.g. it should not increase under the action of CPTP map. As geometric distance between two quantum states can be measured from different aspects, the GQDs can be defined accordingly, provided that they satisfy the conditions for a proper measure of quantum correlation discussed in section 4.4.

4.3.1 Quantum discord through Hilbert-Schmidt norm

The initial measure of GQD was introduced by Dakić, Vedral and Brukner (DAKIĆ; VEDRAL; BRUKNER, 2010). This measure quantifies the amount of quantum correlations of a state in terms of its minimal distance from the set of classical states, where the characterization of the distance between two states is given by the Hilbert-Schmidt (HS) norm, and the GQD of ρ is defined as:

$$D_G(\rho) = \min_{\rho_c \in \mathcal{C}} \| \rho - \rho_c \|_2^2, \tag{4.17}$$

with $\| X \|_2$ being the HS norm which is defined as $\| X \|_2 = \sqrt{\mathrm{tr}(X^\dagger X)}$. A definition equivalent to the GQD is given by (LUO; FU, 2010) :

$$D_G(\rho) = \min_{\Pi^A} \| \rho - \Pi^A \rho \|_2^2, \tag{4.18}$$

where $\Pi^A = \{\Pi_k^A\}$ is the local von Neumann measurements on party A, and $\Pi^A(\rho) = \sum_k (\Pi_k^A \otimes \mathbb{I}_B)\rho(\Pi_k^A \otimes \mathbb{I}_B)$. This implies that the minimization in Eq. (4.18) can be taken only over the subset of classical-quantum states $\{\Pi^A(\rho)\}$. The calculations for this measure requires a simpler minimization procedure, with analitical solutions for general two-qubit states. The HS norm of GQD also plays potential role in remote state preparation (DAKIĆ et al., 2012). But this difinition of GQD cannot be regarded as a good measure for the quantum correlation because it is noncontractive, this is, its value may be changed even under trivial local reversible operations on the unmeasured party B, hence is not well defined (PIANI, 2012).

4.3.2 Quantum discord through trace-norm

The measure of D_G presented in the previous section fails as a rigorous quantifier of correlation. Explicitly, assuming the map $\Gamma^\sigma : X \to X \otimes \sigma$, which represent a channel that introduces a noisy ancillary state, Piani demonstrated in (PIANI, 2012) that $D_G(\Gamma_b^\sigma[\rho]) = D_G(\rho)\mathrm{tr}[\sigma^2]$. This means that the geometric discord may increase under local operations on the unmeasured subsystem b, because $\mathrm{tr}[\sigma^2] \leq 1$ in general. As noted by Paula, de Oliveira, and Sarandy 2013, the origin of this problem is the HS norm, which is not appropriate choice for geometrically quantifying the quantumness of correlations. Therefore, instead of using HS norm was considered the possibility of using the general Schatten p-norm to measure quantum correlations (PAULA; OLIVEIRA; SARANDY, 2013). Let us consider the geometric discord based on a more general norm, defined as (DEBARBA; MACIEL; VIANNA, 2012):

$$D_p(\rho) = \min_{\rho_c \in \mathcal{C}} \| \rho - \rho_c \|_p^p, \tag{4.19}$$

where the Schatten p-norm for a matrix M is given by:

$$\| M \|_p = \{\mathrm{tr}[M^\dagger M]^{p/2}\}^{1/p}, \tag{4.20}$$

which reduces to the HS norm if $p = 2$, and the trace-norm if $p = 1$. Here p is a positive integer number. Using the multiplicative property of the Schatten p-norm under tensor products, it is possible to observe that $\| X \|_p \to \| \Gamma_b^\sigma[X] \|_p = \| X \|_p \| \sigma \|_p$. Hence (PAULA; OLIVEIRA; SARANDY, 2013):

$$D_p(\Gamma_b^\sigma[\rho]) = D_p(\rho) \| \sigma \|_p^p \, . \tag{4.21}$$

Remarkably, $\| \sigma \|_1 = 1$ if and only if $p = 1$, since $\| \sigma \|_1 = \text{tr}[\rho] = 1$ for a general state σ. Therefore, the geometric discord based on the 1-norm is the only possible Schatten p-norm able to consistently quantify non-classsical correlations.

In this order, the trace-norm of discord (TND) or 1-norm geometric quantum discord is defined as (PAULA; OLIVEIRA; SARANDY, 2013):

$$D_1(\rho) = \min_{\rho_c \in \mathcal{C}} \| \rho - \rho_c \|_1, \tag{4.22}$$

where $\| X \|_1 = \text{tr}[\sqrt{X^\dagger X}]$ is the 1-norm and for $2 \times n$ dimensional state ρ (i.e. A is a qubit), the optimal ρ_c can also be obtained from the subset $\Pi^A(\rho)$, with $\Pi^A = \{\Pi_k^A\}$ being the set of local projective measurements, i.e., $D_1(\rho) = \min_{\Pi^A} \| \rho - \Pi^A(\rho) \|_1$ (NAKANO; PIANI; ADESSO, 2013).

The calculation of $D_1(\rho)$ is a hard task, and there is no analytical solution for it in general cases. In the following, we will describe class of states where this measure can be calculated explicitly.

4.3.2.1 Geometric one-norm quantum correlations for two-qubit Bell diagonal states

We will focus here in the particular case of two-qubit Bell diagonal states, this class of two-qubit states are diagonal in the Bell basis (DAKIĆ; VEDRAL; BRUKNER, 2010; DAKIĆ et al., 2012; LANG; CAVES, 2010). The Bell-diagonal states of two-qubits, A and B, have density operators of the form[5]:

$$\rho^{Bell} = \frac{1}{4}[\mathbb{I} \otimes \mathbb{I} + \vec{c}.(\vec{\sigma} \otimes \vec{\sigma})] = \sum_{ab} \lambda_{ab} |\beta_{ab}\rangle\langle\beta_{ab}|, \tag{4.23}$$

with the corresponding density matrix of ρ^{Bell} to be

$$\rho^{Bell} = \frac{1}{4} \begin{pmatrix} 1+c_3 & 0 & 0 & c_1 - c_2 \\ 0 & 1-c_3 & c_1+c_2 & 0 \\ 0 & c_1+c_2 & 1-c_3 & 0 \\ c_1 - c_2 & 0 & 0 & 1+c_3 \end{pmatrix}, \tag{4.24}$$

where $\mathbb{I}$ is the identity matrix on the subsystem, $\vec{c} = (c_1, c_2, c_3)$ is a three-dimensional vector and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is a vector formed by Pauli matrices. The eigenstates of ρ^{Bell} are the four Bell states (LANG; CAVES, 2010):

$$|\beta_{ab}\rangle \equiv (|0, b\rangle + (-1)^a |1, 1 \oplus b\rangle)/\sqrt{2}, \tag{4.25}$$

[5] Here we use a computational base $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

with eigenvalues

$$\lambda_{ab} = \frac{1}{4}[1 + (-1)^a c_1 - (-1)^{a+b} c_2 + (-1)^b c_3], \tag{4.26}$$

where $a \in \{0, 1\}$, $b \in \{0, 1\}$. Any two-qubits state satisfying $\langle \sigma_j^A \rangle = 0 = \langle \sigma_j^B \rangle$, i.e, having maximally mixed marginal density operators $\rho_A = \mathbb{I}/2 = \rho_B$, can be brought to Bell-diagonal form by using local unitary operations on the two-qubits to diagonalize the correlation matrix $\langle \sigma_j^A \otimes \sigma_k^B \rangle$.

A Bell-diagonal state is specified by a 3-tuple (c_1, c_2, c_3). The density operator must be a positive operator, then $0 \leq \lambda_{ab} \leq 1$ and $\sum_{i,j} \lambda_{i,j} = 1$. The resulting region of Bell-diagonal states is the state tetrahedron $\mathcal{T}$ in Fig. 5, whose vertices situated on the points $(1, 1, -1), (-1, -1, -1), (1, -1, 1)$ and $(-1, 1, 1)$ represent the Bell states.

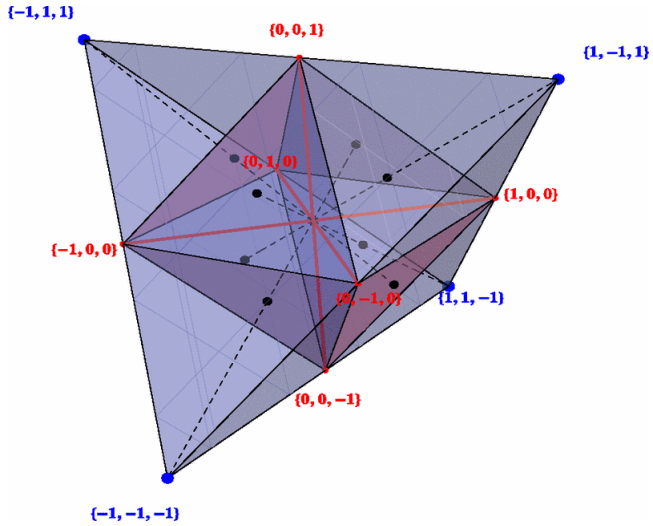

Figure 5 – Two-qubit Bell-diagonal states described by parameters c_1, c_2, c_3 can be depicted as a tetrahedron $\mathcal{T}$. The octahedron $\mathcal{O}$ in tetrahedron, is the set of separable Bell-diagonal states. The zero-discord states Eq. (4.15) are labeled by the red lines (over the perpendicular axis c_1, c_2 and c_3). Quantum discord is a maximum ($D = 1$ and $D_G = 1/2$) in the vertices of the tetrahedron corresponding to the four Bell states. Among the set of separable states, those which maximize discord are the centers of octahedron facets $(\pm 1, \pm 1, \pm 1)/3$ (black dots). This figure is reproduced from (DAKIĆ; VEDRAL; BRUKNER, 2010).

Assuming that the minimal state preserves the Bell-diagonal form of the original state and following the line of thought described in Appendix C, the 1-norm geometric quantum discord two-qubit Bell diagonal states is given by (PAULA; OLIVEIRA; SARANDY, 2013):

$$D_T(\rho^{Bell}) = \text{int}\{|c_1|, |c_2|, |c_3|\}, \tag{4.27}$$

with int$\{\cdot\}$ denoting the intermediate among the elements of the set $\{|c_1|, |c_2|, |c_3|\}$.

In order to determine the geometric and total correlations for Bell-diagonal states, a geometric point of view of the correlations through the trace distance was used in (PAULA et al., 2014). Therefore, the geometric quantum, classical and total correlations can be defined as follows (see Fig. 6):

$$D_T = \| \rho - \mathcal{M}^A(\rho) \|_1 = \mathrm{tr}|\rho - \mathcal{M}^A(\rho)|, \tag{4.28}$$

$$C_T = \| \mathcal{M}^A(\rho) - \mathcal{M}^A(\pi_\rho) \|_1 = \mathrm{tr}|\mathcal{M}^A(\rho) - \mathcal{M}^A(\pi_\rho)|, \tag{4.29}$$

$$T_T = \| \rho - \pi_\rho \|_1 = \mathrm{tr}|\rho - \pi_\rho|, \tag{4.30}$$

where a state ρ is given by Eq. (4.23), π_ρ represent the product of the local marginals of ρ (for two-qubits $\pi_\rho = \rho_A \otimes \rho_B$) and $\mathcal{M}^A(\rho)$ is the classical state that emerge from a projective measurement on subsystem A that minimizes Eq. (4.28). In this case, the classical state $\mathcal{M}^A(\rho)$ is defined as follows (PAULA et al., 2014):

$$\mathcal{M}^A(\rho) = \sum_{k=-,+} (\Pi_k \otimes \mathbb{I})\rho(\Pi_k \otimes \mathbb{I}), \tag{4.31}$$

where

$$\Pi_\pm = \frac{1}{2}(\mathbb{I} \pm \vec{n}.\vec{\sigma}), \tag{4.32}$$

are the projection operators, $\mathbb{I}$ is the identity matrix, $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is a vector formed by Pauli matrices, and $\vec{n} = (n_1, n_2, n_3)$ is a unitary vector that minimizes D_T[6].

In Appendix D we recall the method described in (PAULA et al., 2014) to find the unitary vector that minimizes D_T. Hence, the geometric classical correlation C_T and total correlation T_T as for the Bell diagonal states are given by:

$$C_T(\rho^{Bell}) = c_+, \tag{4.33}$$

$$T_T(\rho^{Bell}) = \frac{1}{2}[c_+ + \max\{c_+, c_0 + c_-\}], \tag{4.34}$$

with c_+, c_- and c_0 being the maximum, minimum, and intermediate values of $\{|c_1|, |c_2|, |c_3|\}$, respectively.

[6] Remarkably, the authors in (PAULA et al., 2014) defined the geometric correlations by fixing $\rho_c \in \{\mathcal{M}^A(\rho)\}$ and denoting $\Pi_\pm$ the corresponding optimal measurement operator for obtained $D_T(\rho)$ (the minimization over $\{\mathcal{M}^A(\rho)\}$ is equivalent to the minimization over $\mathcal{C}$ for qubit states, Eq. (4.22)).

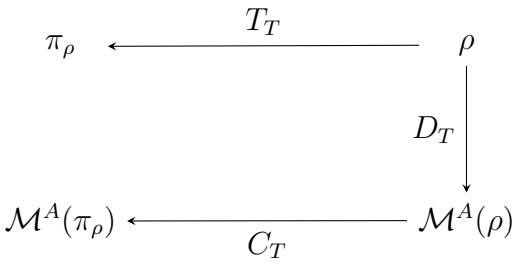

Figure 6 – Geometric correlations in the state ρ, where $\mathcal{M}^A$ is the optimal measurement operator for obtaining D_T, $\pi_\rho = \rho_A \otimes \rho_B$.

An important point to highlight here is that the generalization of the Bell-diagonal states is given by the X states (YU; EBERLY, 2007). This set of states, includes maximally entangled Bell states and separable and non separable Werner states (WERNER, 1989) as special cases. The quantum contribution for the geometric correlation for X states has been analytically derived (CICCARELLO; TUFARELLI; GIOVANNETTI, 2014). Moreover, a closed analytical expressions for the geometric classical and geometric total correlations are our contribution (OBANDO; PAULA; SARANDY, 2015). Therefore, in the subsequent chapter we are going to show the description of some properties of X states and how calculate the classical and total correlations for X states.

4.4 Quantum, classical and total correlation

There exist in the literature a variety of fundamental concepts to define the quantum correlations measures. For example, the information gain from a measurement, the effects of measurements on a system, the notion of classical states, and the lack of correlations in product states. Hence, one of the principal question is if a measure which is associated with a certain physical scenario is indeed a measure of correlation (BRODUTCH; MODI, 2012). This section provides an overview of the set of criteria for a proper measure of correlations.

4.4.1 Criteria for classical correlations

Based on the criteria for entanglement measures, Henderson and Vedral (HENDERSON; VEDRAL, 2001) proposed the following set of criteria that a measure of classical correlations must be satisfy:

1. Product states are uncorrelated.
2. Classical correlations are invariant under local-unitary operations.
3. Classical correlations are nonincreasing under local operations.

4. Similarly that in the quantum case, the classical correlations for pure states are given by the entropy of the reduced states. They show that their measure $J(A|B)$ satisfies these axioms.

4.4.2 General properties for correlations

Brodutch and Modi (BRODUTCH; MODI, 2012) present a more general method for construing correlation measures (for bipartite and multipartite systems). This method lead to a set of criteria for measures of correlations which can be divided into tree categories: (1) necessary conditions, (2) reasonable properties, and (3) debatable criteria.

Therefore, the correlations are measured using a positive-real valued function and a set local measurements.

Definition 1. The generalized-discord function $\mathcal{K}[\rho_1, \rho_2]$ is defined over all quantum states ρ_1 and ρ_2 with properties $\mathcal{K}[\rho_1, \rho_2] \in \mathbf{R}^+$ and $\mathcal{K}[\rho, \rho] = 0$.

Definition 2. The set of measurements $\{\mathcal{M}\}$ is a set of general quantum-operations that are trace-preserving.

Definition 3. $\mathcal{M}(\rho)$ is a classically correlated state. Frequently $\{\mathcal{M}\}$ is a full set of POVM on one or more of the subsystems with or without communication. The measurement $\mathcal{M}_\rho$ depend on the quantum state ρ.

In this way, using the generalized-discord function $\mathcal{K}$ is possible to define classical, quantum and total correlations (BRODUTCH; MODI, 2012):

- The generalized quantum discord (quantum correlations) of a state ρ is defined as the "distance"[7] between the state ρ and the classical state $\mathcal{M}(\rho)$:

$$\mathcal{Q}(\rho) \equiv \mathcal{K}[\rho, \mathcal{M}_\rho(\rho)], \tag{4.35}$$

  where $\mathcal{M}(\rho)$ minimize the quantum correlations.

- Classical correlations of a state ρ are defined as the distance between the classical states $\mathcal{M}(\rho)$ and the reduced product state after the same measurement $\mathcal{M}(\pi_\rho)$:

$$\mathcal{C}(\rho) \equiv \mathcal{K}[\mathcal{M}(\rho), \mathcal{M}(\pi_\rho)], \tag{4.36}$$

  here $\mathcal{M}(\rho)$ maximize classical correlations and $\mathcal{M}(\pi_\rho) \equiv \otimes_i[\mathcal{M}(\rho)]^i$ where the marginals are given by $\pi_\rho = \otimes_i \rho^i$ with $\rho^i = \mathrm{tr}_i \rho$.

[7] $\mathcal{K}$ may not be a proper distance.

- The total correlations of a state ρ are defined as the distance between that state ρ and the reduced product π_ρ.

$$\mathcal{T} \equiv \mathcal{K}[\rho, \pi_\rho] \tag{4.37}$$

where π_ρ is the product of the marginals of ρ, $\pi_\rho = \otimes_i \rho^i$.

The five necessary conditions for any measure of correlations are then:

1) Product states have no correlations: $\mathcal{T}(\pi) = \mathcal{Q}(\pi) = \mathcal{C}(\pi) = 0$

2) All correlations are invariant under local unitary operations.

3) All correlations are non-negative: $\mathcal{C} \geq 0$ and $\mathcal{Q} \geq 0$ and $\mathcal{T} \geq 0$.

4) $\mathcal{T}$ is non-increasing under local operations.

5) Classical states have no quantum correlations. $\mathcal{Q}(\mathcal{M}_\chi(\rho)) = 0$ for all ρ and $\mathcal{M}_\chi$.[8]

The next three criteria involving continuity, described as reasonable are

a) Continuity under small perturbations.

b) Strong continuity of measurement basis under small perturbations.

c) Weak continuity of the measurement basis under small perturbations.

Finally a set of criteria based on entanglement measures and information-theoretical ideas are presented as debatable:

i) For pure bipartite states the correlations can be defined by the marginals.

ii) Correlations are additive $\mathcal{T} = \mathcal{C} + \mathcal{Q}$ or superadditive $\mathcal{T} < \mathcal{C} + \mathcal{Q}$.

iii) Classical and/or quantum correlations are nonincreasing under local operations.

iv) Symmetry under the interchange of subsystems.

Of these criteria, the five identified as more fundamental define if the physical quantity can or cannot be consider a proper measure of correlation. The correlations measures presented in this thesis are all consistent whit these requirements. On the other hand in the second set of criteria, strong continuity means that the measurement which minimizes discord is changed continuously for small changes in the state, and weak

[8] The measurement is the same for all ρ so $\{M\} = M_\chi$, the subscript χ represents a fixes basis.

continuity means that the measurement is not necessarily continuous but using the basis which optimizes a nearby state result in a small error for calculating the correlations. The measures based on measurements which do not affect the marginals (MID) are found to fail all of the above continuously requirements. However, the measures of discord with $\mathcal{M}_\rho$ that minimize the quantum correlations such as discord and the geometric discord are proved to be continuous. This include all measures that will be present in the next chapters. Finally, the relevance for the set of criteria debatable is leave open for the authors in (BRODUTCH; MODI, 2012), although most measures of correlations meet most of these criteria.

4.5 Quantum correlations and non-Markovianity: RHP measure

In the previous Chapter, we showed the approach the BLP measure, which is based in the idea that memory effects in dynamics of open quantum systems are linked to the exchange of information between the open system and its environment. BLP defines non-Markovian dynamics as an increase in the distinguishability of pairs of evolving quantum states. In this section, our aim is to discuss an alternative measure of non-Markovianity introduced by (RIVAS; HUELGA; PLENIO, 2010), this measure, known as RHP measure, is based on the specific behavior of quantum correlations when a part of a composite system is subject to a local interaction that can be modeled as a CPTP map.

In the approach of RHP, it is proposed a strategy to quantify the non-Markovian character of quantum evolutions by a measure denoted by $\mathcal{I}^{\mathcal{E}}$ that quantifies the deviation from Markovianity in the evolution of the system. Thus, the non-Markovianity can be quantified using the specific behavior of quantum correlations when a part of a composite system is subject to a local interaction that can be modeled through CPTP maps (RIVAS; HUELGA; PLENIO, 2014). To establish the measure, it is initially considered a maximally entangled state with an ancillary system which has to remain isolated from the decoherence sources, as showed in Fig. 7. Considering the fact that entanglement does not increase under local operations (PLENIO; VIRMANI, 2005), it is immediate from the CP-divisibility (2.44) that the decay of the entanglement with an ancillary system will be monotonically decreasing for Markovian evolutions. Nevertheless, the requirement of strict monotonicity has no validity for a non-Markovian evolution, environmental correlations may lead to bipartite entanglement increase at certain times, this behavior is illustrated by the red curve in Fig. 7. From this, the quantifier of the degree of non-Markovianity of a quantum evolution can be obtained from the amount of entanglement between system (S) and ancilla (A) at different times within a selected interval $[t_0, t_{max}]$. Considering the initial maximally entangled system-ancilla state described by $\rho_{SA}(0) = |\Phi\rangle\langle\Phi|$ with $|\Phi\rangle = \frac{1}{\sqrt{d}}\sum_{n=0}^{d-1}|n\rangle|n\rangle$.

The variation $\Delta\mathcal{E}$ of entanglement between the final and the initial state is

$$\Delta\mathcal{E} = \mathcal{E}[\rho_{SA}(t_0)] - \mathcal{E}[\rho_{SA}(t_{max})], \tag{4.38}$$

where $\mathcal{E}$ is some entanglement measure. Then, non-Markovianity is quantified by (RIVAS; HUELGA; PLENIO, 2014):

$$\mathcal{I}^{(\mathcal{E})} = \int_{t_0}^{t_{max}} \left| \frac{d\mathcal{E}(\rho_{SA}(t))}{dt} \right| dt - \Delta\mathcal{E}. \tag{4.39}$$

It then follows that if the evolution of the system is Markovian the derivative of $\mathcal{E}(\rho_{SA}(t))$ is always negative and $\mathcal{I}^{(\mathcal{E})} = 0$.

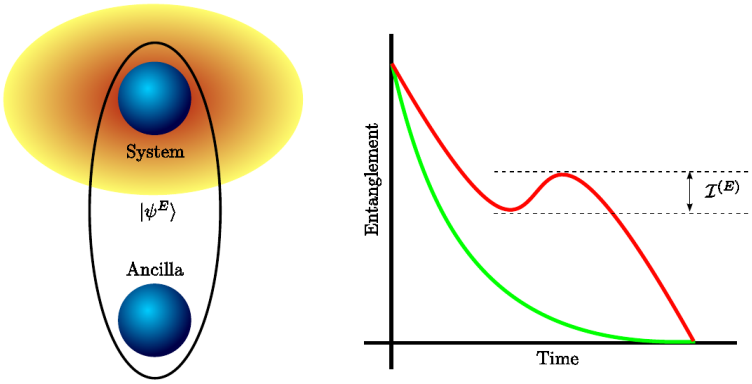

Figure 7 – Schematic illustration of the non-Markovianity measure proposed by Rivas et al. (RIVAS; HUELGA; PLENIO, 2014). An arbitrary quantum system is subject to the action of a local bath, depicted as a golden shadow. The system is initially prepared in a maximally entangled state Φ with an ancilla. When starts the evolution of the system-ancilla, the entanglement will be sensitive to the environment coupling. The green line represents some typical decay of the initial entanglement for a Markovian evolution while the red line corresponds to a possible non-Markovian decay. Here, $\mathcal{I}(\mathcal{E})$ allows us to estimate the non-Markovianity of the process despite. This figure is reproduced from (RIVAS; HUELGA; PLENIO, 2014).

The RHP measure gives a connection between the entanglement and the non-Markovianity though a meaning in terms of information flow is still lacking in this approach. Entanglement-based measures consider the amount of deviation from divisibility of a given dynamical map, where any measure of entanglement monotonically decreases for all divisible processes (RIVAS; HUELGA; PLENIO, 2014). The entanglement-based measure proposed by (FANCHINI et al., 2014) have a direct information based interpretation. For this method it is considered the overall increase of $\mathcal{E}_{SA}$ along the whole time evolution and it includes an optimization measure. Therefore, this measure is given by (FANCHINI et al., 2014):

$$\mathcal{N}(\Phi) \equiv \max_{\rho_{SA}(0)} \int_{\frac{d\mathcal{E}_{SA}}{dt}>0} \frac{d}{dt}\mathcal{E}_{SA}(t)dt, \tag{4.40}$$

where the maximization is taken over all possible pure initial states of the bipartite system SA. Assuming that the amount of entanglement is quantified by the entanglement of formation (EOF). The EOF denoted $\mathcal{E}_f$ is defined by (BENNETT et al., 1996):

$$\mathcal{E}_f(\rho_{AB}) = \min_{\{p_i,|\phi_i\rangle\}} \sum_i p_i S(\mathrm{tr}_B[|\phi_i\rangle\langle\phi_i|]), \tag{4.41}$$

where $S(\rho)$ is the von Neumann entropy of ρ Eq.(4.7), and the minimum is taken over all ensembles $\{p_i, |\phi_i\rangle\}$ satisfying $\sum_i p_i|\phi_i\rangle\langle\phi_i| = \rho_{AB}$. Considering that the initial state of the environment E is pure. Thus, the tripartite state SAE is pure, the Koashi-Winter relation gives (KOASHI; WINTER, 2004):

$$\mathcal{E}_f\rho(SA) = S(\rho_S) - J^{\leftarrow}_{SE}, \tag{4.42}$$

where $J^{\leftarrow}_{SE}$ denotes the maximum amount of classical information Eq. (4.6), that can be extracted about the system by the observation of the environment E and is defined by

$$J^{\leftarrow}_{SE} = \max_{\{\Pi^E_i\}}[S(\rho_S) - \sum_i p_i S(\rho_{S|i})], \tag{4.43}$$

where $S(\rho)$ is the von Neumann entropy Eq.(4.7) and $\{\Pi^E_i\}$ represents the general quantum measures acting on the environment E. The term $\rho_{S|i}$ is given by Eq.(4.9) and stand for the remaining state of the subsystem S after obtaining the outcome i with probability p_i in the subsystem E. A time derivative of Eq. (4.42), yields

$$\frac{d}{dt}\mathcal{E}_{SA} = -\frac{d}{dt}J^{\leftarrow}_{SE}, \tag{4.44}$$

where ρ_S is time invariant, due to S does not directly interact with E. It is immediate to see from the above equation that any temporary increase in $\mathcal{E}_{SA}$ during the dynamics of the open system, implies a temporary decrease in $J^{\leftarrow}_{SE}$. Thus any deviation from the monotonically increasing behavior of $J^{\leftarrow}_{SE}$ is an indication of non-Markovianity. This connection presents the relation between entanglement-base measure and the flow of information between the system and the environment in terms of an entropic measure of information (FANCHINI et al., 2014).

We will present in Chapter 6, our contributions to the characterization non-Markovianity of a quantum system. In this line of thought, we propose a general framework to characterize non-Markovianity through multipartite measures of quantum, classical and total correlations.

5 TRACE-DISTANCE CORRELATIONS FOR X STATES AND EMERGENCE OF THE POINTER BASIS IN MARKOVIAN AND NON-MARKOVIAN REGIMES

In this Chapter, we develop a method to evaluate both classical and total trace-norm (Schatten 1-norm) geometric correlations for the complete set of X states. As we mentioned in section 4.3, for the simple case of mixed two-qubit systems in Bell-diagonal states, analytical expressions have been found for quantum, classical, and total correlations. However, for the more general case of two-qubit X states described in section 5.1, only the quantum contribution for the geometric correlation has been analytically derived (CICCARELLO; TUFARELLI; GIOVANNETTI, 2014).

Therefore, our aim is to close this gap, providing in section 5.2, closed analytical expressions for the classical and total correlations of arbitrary two-qubit X states. The analytical expressions for the classical correlation of X states can be applied as a powerful resource to characterize the open-system dynamics in rather general environments. As an application, in section 5.3 we consider the open-system dynamical behavior of such correlations under PD and GAD evolutions. Finally, we show that geometric classical correlations can characterize the emergence of the pointer basis of an apparatus subject to decoherence in either Markovian or non-Markovian regimes.

5.1 X states

We are interested in the evolution of classical and total correlations under noise of an important two-qubit system known as " X states". The density matrix of these states only contains non-zero elements in an "X" formation, along the main diagonal and anti-diagonal. Thus the density matrix shows the general form:

$$\rho_X = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{41}^* \\ 0 & \rho_{22} & \rho_{32}^* & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}, \tag{5.1}$$

where the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ is adopted. This visual appearance resembling that the letter X has led them to be called " X states" (YU; EBERLY, 2007).

Eq. (5.1) describes a quantum state provided the unit trace and positivity conditions $\sum_{i=1}^{4}\rho_{ii}=1$, $\rho_{11}\rho_{44}\geq|\rho_{41}|^2$ and $\rho_{22}\rho_{33}\geq|\rho_{32}|^2$ are fulfilled. The diagonal elements are real, whereas the elements ρ_{41} and ρ_{32} are complex numbers in general. However, they can be brought into real numbers via local unitary transformations, which preserve the trace distance correlations (CICCARELLO; TUFARELLI; GIOVANNETTI, 2014).

We particularly note that these states generalize many important classes of mixed quantum states such as the maximally entangled Bell states (NIELSEN; CHUANG, 2000), partially entangled and quantum correlated states (like the well-known Werner mixed state (WERNER, 1989)), lending further importance to their study. The X states were first identified as a class of states of interest in the work of Yu and Eberly (YU; EBERLY, 2007), where some of their properties in connection with the phenomenon of sudden death of entanglement were investigated. The properties of this states were studied for example in (ALI; RAU; ALBER, 2010; MAN'KO; MARKOVICH, 2014). The X mixed states arises naturally in a wide variety of physical situations (WANG et al., 2006; PRATT, 2004; BOSE et al., 2001) and frequently were encountered in condensed matter systems, quantum dynamics, etc, (DILLENSCHNEIDER, 2008; LANG; CAVES, 2010; WERLANG et al., 2010; CILIBERTI; ROSSIGNOLI; CANOSA, 2010). Remarkably, the mixed states defined here not only are rather common but also have the property that they often retain the X form under noisy evolution.

An equivalent and rather useful way of writing the states (5.1) is by decomposing in terms of Pauli operators Eq. (2.22):

$$\rho_X=\frac{1}{4}\left(\mathbb{I}\otimes\mathbb{I}+\sum_{i=1}^{3}c_i\sigma_i\otimes\sigma_i+c_4\mathbb{I}\otimes\sigma_3+c_5\sigma_3\otimes\mathbb{I}\right), \tag{5.2}$$

the elements of the density matrix and the parameters c_i can be related as follows:

$$c_1=\mathrm{tr}(\sigma_1\otimes\sigma_1\rho_X)=2(\rho_{32}+\rho_{41}), \tag{5.3}$$

$$c_2=\mathrm{tr}(\sigma_2\otimes\sigma_2\rho_X)=2(\rho_{32}-\rho_{41}), \tag{5.4}$$

$$c_3=\mathrm{tr}(\sigma_3\otimes\sigma_3\rho_X)=1-2(\rho_{22}+\rho_{33}), \tag{5.5}$$

$$c_4=\mathrm{tr}(\mathbb{I}\otimes\sigma_3\rho_X)=2(\rho_{11}+\rho_{33})-1, \tag{5.6}$$

$$c_5=\mathrm{tr}(\sigma_3\otimes\mathbb{I}\rho_X)=2(\rho_{11}+\rho_{22})-1, \tag{5.7}$$

where $\mathbb{I}$ is the identity matrix and the parameters c_i assuming values in the interval $-1 \leq c_i \leq 1$. Finally, we note that for the special case of two-qubit X states with $c_4 = c_5 = 0$, the Bell diagonal states described in subsection 4.3.2.1 are recovered.

5.2 Geometric classical and total correlations: analytical expressions

In this section we provide analytical expressions for geometrical classical and total correlations for two-qubits X states. As mentioned beforehand in Chapter 4, particularly section 4.4.2, the general approach of measures of quantum, classical, and total correlations of an n-partite system in a state ρ are respectively defined by (MODI et al., 2011; BRODUTCH; MODI, 2012; PAULA et al., 2014):

$$Q(\rho) = \mathcal{K}\left[\rho, M_-(\rho)\right], \tag{5.8}$$

$$C(\rho) = \mathcal{K}\left[M_+(\rho), M_+(\pi_\rho)\right], \tag{5.9}$$

$$T(\rho) = \mathcal{K}\left[\rho, \pi_\rho\right], \tag{5.10}$$

where $\mathcal{K}\left[\rho, \tau\right]$ denotes a real and positive function that vanishes for $\rho = \tau$, $M_-(\rho)$ is a classical state obtained through a non-selective measurement $\{M_-^{(i)}\}$ that minimizes Q, $M_+(\rho)$ is a classical state obtained through a non-selective measurement $\{M_+^{(i)}\}$ that maximizes C, and the product of the local marginals of ρ are given by $\pi_\rho = \rho_1 \otimes ... \otimes \rho_n = \mathrm{tr}_{\bar{1}}\rho \otimes ... \otimes \mathrm{tr}_{\bar{n}}\rho$. In order to avoid ambiguities in the correlation measures for Q and C, we take $\{M_-^{(i)}\}$ and $\{M_+^{(i)}\}$ as independent measurement sets (PAULA et al., 2014).

For our purpose we consider correlations based on the trace-norm (Schatten 1-norm) and projective measurements operating over one qubit of a two-qubit system, i.e., $\mathcal{K}\left[\rho, \tau\right] = \|\rho - \tau\|_1 = \mathrm{tr}|\rho - \tau|$ and $M_\pm(\rho) = \Pi_\pm^{(1)}(\rho)$, such that

$$Q_G(\rho) = \mathrm{tr}\left|\rho - \Pi_-^{(1)}(\rho)\right|, \tag{5.11}$$

$$C_G(\rho) = \mathrm{tr}\left|\Pi_+^{(1)}(\rho) - \Pi_+^{(1)}(\pi_\rho)\right|, \tag{5.12}$$

$$T_G(\rho) = \mathrm{tr}\left|\rho - \pi_\rho\right|. \tag{5.13}$$

Q_G is known like Schatten 1-norm geometric quantum discord and was introduced in (PAULA; OLIVEIRA; SARANDY, 2013; NAKANO; PIANI; ADESSO, 2013). As we already discussed in section 4.3, the counterpart to Q_G, C_G is the Schatten 1-norm classical

correlation. Concerning T_G, it is a measure of total geometric correlation, which vanishes if the system is described by a product state. The trace-norm satisfies reasonable criteria expected for correlation measures, although these criteria are still source of debate (PAULA et al., 2014; MAZIERO, 2015). Recalling that the analytic formula for Schatten 1-norm quantum correlation for X states has been shown in (CICCARELLO; TUFARELLI; GIOVANNETTI, 2014). It is given in terms of the parameters $\{c_i\}$ by:

$$Q_G(\rho_X) = \sqrt{\frac{ac - bd}{a - b + c - d}}, \tag{5.14}$$

where $a = \max\{c_3^2, d + c_5^2\}$, $b = \min\{c, c_3^2\}$, $c = \max\{c_1^2, c_2^2\}$, and $d = \min\{c_1^2, c_2^2\}$. In order to present our construction for the corresponding classical and total correlations, let us first compute the marginal density operators:

$$\rho_1 = \mathrm{tr}_{\bar{1}}\rho = \frac{1}{2}\left(\mathbb{I} + c_5\sigma_3\right) \text{ and } \rho_2 = \mathrm{tr}_{\bar{2}}\rho = \frac{1}{2}\left(\mathbb{I} + c_4\sigma_3\right). \tag{5.15}$$

Thus, the product state $\pi_{\rho_X} = \rho_1 \otimes \rho_2$ reads

$$\pi_{\rho_X} = \frac{1}{4}\left(\mathbb{I}\otimes\mathbb{I} + c_4\mathbb{I}\otimes\sigma_3 + c_5\sigma_3\otimes\mathbb{I} + c_4c_5\sigma_3\otimes\sigma_3\right). \tag{5.16}$$

An interesting point is that the inner expression of classical correlation given by Eq. (5.12) can be rewritten as:

$$\Pi_+^{(1)}(\rho) - \Pi_+^{(1)}(\pi_\rho) = \Pi_+^{(1)}(\rho - \pi_\rho). \tag{5.17}$$

As can be seen from Eqs. (5.2) and (5.16), the difference of X states $\rho_X - \pi_{\rho_X}$ is mathematically equivalent to a difference between Bell-diagonal states. This fact allow us to rewrite the difference $\rho_X - \pi_{\rho_X}$ in terms of effective Bell-diagonal states $\tilde{\rho}_B$ and $\pi_{\tilde{\rho}_B}$ as follows:

$$\rho_X - \pi_{\rho_X} = \tilde{\rho}_B - \pi_{\tilde{\rho}_B}, \tag{5.18}$$

where

$$\tilde{\rho}_B = \frac{1}{4}\left[\mathbb{I}\otimes\mathbb{I} + \sum_{i=1}^{3}\tilde{c}_i\sigma_i\otimes\sigma_i\right] \tag{5.19}$$

and

$$\pi_{\tilde{\rho}_B} = \frac{1}{4}\left(\mathbb{I}\otimes\mathbb{I}\right), \tag{5.20}$$

with

$$(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3) = (c_1, c_2, c_3 - c_4c_5). \tag{5.21}$$

We now apply the result obtained above for evaluating the analytical expressions of C_G and T_G, using the Eq. (4.33) and Eq. (4.34) presented in section 4.3 for the Bell-diagonal

states. As a consequence, the geometrical classical and total correlations for X states are given by:

$$C_G(\rho_X) = C_G(\tilde{\rho}_B) = \tilde{c}_+, \tag{5.22}$$

and

$$T_G(\rho_X) = T_G(\tilde{\rho}_B) = \frac{1}{2}\left[\tilde{c}_+ + \max\{\tilde{c}_+, \tilde{c}_0 + \tilde{c}_-\}\right], \tag{5.23}$$

where $\tilde{c}_- = \min\{|\tilde{c}_1|, |\tilde{c}_2|, |\tilde{c}_3|\}$, $\tilde{c}_0 = \text{int}\{|\tilde{c}_1|, |\tilde{c}_2|, |\tilde{c}_3|\}$, and $\tilde{c}_+ = \max\{|\tilde{c}_1|, |\tilde{c}_2|, |\tilde{c}_3|\}$ represent the minimum, intermediate, and the maximum of the absolute values of the parameters $\tilde{c}_i$ $(i = 1, 2, 3)$, respectively.

5.3 Applications

In this section, we will illustrate the applicability of the geometric measure of classical correlations by considering the decohering dynamics of the quantum systems. We will take the system as a X state coupled independently with sources of Markovian and non-Markovin noises. In particular, we centered our attention on PD and GAD noises.

5.3.1 Markovian dynamics

Having in mind the operator-sum representation formalism presented in section 2.4. We consider qubits A and B prepared in a mixed state ρ. The Markovian evolution of ρ is governed by a trace-preserving quantum operation, which can be written compactly as

$$\varepsilon(\rho) = \sum_{i,j} \left(K_i^A \otimes K_j^B\right) \rho \left(K_i^A \otimes K_j^B\right)^\dagger, \tag{5.24}$$

where the Kraus operators $\{K_k^s\}$ satisfy the trace-preserving condition reading $\sum_k K_k^{s\dagger} K_k^s = \mathbb{I}$. In the following of this section, we will focus on the PD and GAD channels. We provide in Table 1 the Kraus operators for these channels.

An interesting point is that both the PD and GAD decoherence processes preserve the X form of the density operator. This can be expected for PD noise, which can only give time dependence to the off-diagonal matrix elements. The interaction of the system and GAD noise evolutions are different, and evolution under amplitude noise is more elaborate, affecting all six non-zero elements. In these cases, we check the robust form invariance during evolution.

In order to describe the dynamic evolution of correlations, we have to find out the evolved parameters $\tilde{c}_i(t)$, as defined by Eq. (5.21), inserting the X state Eq. (5.2) into Eq. (5.24). Remarkably, the parameters $\tilde{c}_i(t)$ turn out to be independent of λ_s. Since the evolution is Markovian, we further take the decoherence probability $p_s = 1 - \exp(-t\,\gamma_s)$

	Kraus operators
PD	$K_0^s = \sqrt{1-p_s/2}\, I, K_1^s = \sqrt{p_s/2}\, \sigma_3$
GAD	$K_0^s = \sqrt{\lambda_s}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p_s} \end{pmatrix}, K_2^s = \sqrt{1-\lambda_s}\begin{pmatrix} \sqrt{1-p_s} & 0 \\ 0 & 1 \end{pmatrix}$ $K_1^s = \sqrt{\lambda_s}\begin{pmatrix} 0 & \sqrt{p_s} \\ 0 & 0 \end{pmatrix}, K_3^s = \sqrt{1-\lambda_s}\begin{pmatrix} 0 & 0 \\ \sqrt{p_s} & 0 \end{pmatrix}$

Table 1 – Kraus operators for PD and GAD, where p_s and λ_s are the decoherence probabilities for the qubit s.

for both PD and GAD channels. Therefore, evolution of the parameters of the X state evolution is described in Table 2 in terms of the decoherence time:

$$\tau_D = \frac{1}{\gamma_A + \gamma_B}\,. \tag{5.25}$$

Channel	$\tilde{c}_1(t)$	$\tilde{c}_2(t)$	$\tilde{c}_3(t)$
PD	$c_1 exp[-t/\tau_D]$	$c_2 exp[-t/\tau_D]$	$(c_3 - c_4 c_5)$
GAD	$c_1 exp[-t/2\tau_D]$	$c_2 exp[-t/2\tau_D]$	$(c_3 - c_4 c_5) exp[-t/\tau_D]$

Table 2 – Correlation parameters $\tilde{c}_i(t)$ $(i = 1, 2, 3)$ for PD and GAD channels.

Then, we can directly obtain the dynamics of classical correlations $C_G(\rho_X(t))$, as given by Eq. (5.22). By looking at the correlation parameters $\tilde{c}_i(t)$ from Table 2 one sees immediately that both $|c_1(t)|$ and $|c_2(t)|$ display the same decay rate, which means that they do not cross as functions of time. Therefore, for both PD and GAD channels, only the crossings allowed are for $|c_1(t)| = |c_3(t)|$ and $|c_2(t)| = |c_3(t)|$, implying at most a single nonanalyticity (sudden change) in the geometric classical correlation.

Indeed, a necessary and sufficient condition for sudden change in the case of PD and GAD channels are $\tilde{c}_- = |\tilde{c}_3| \neq 0$ and $\tilde{c}_+ = |\tilde{c}_3| \neq 0$, respectively. Therefore, the generalization of the initial state to an X state does not allow for further sudden changes in the classical correlation. This sustains the result that double sudden changes is an exclusive feature of quantum correlations, as discussed for Bell-diagonal states in Ref. (PAULA et al., 2013a; MONTEALEGRE et al., 2013).

The results are shown in Fig. 8, where we plot C_G as a function of the dimensionless time $\tau = (\gamma_A + \gamma_B)\, t$ for a mixed X state under the GAD channel. In particular we note that a single sudden transition occurs at $\tau_1^* = 0.37$, which can be determined from the correlation parameters $c_i(t)$ in Table 2.

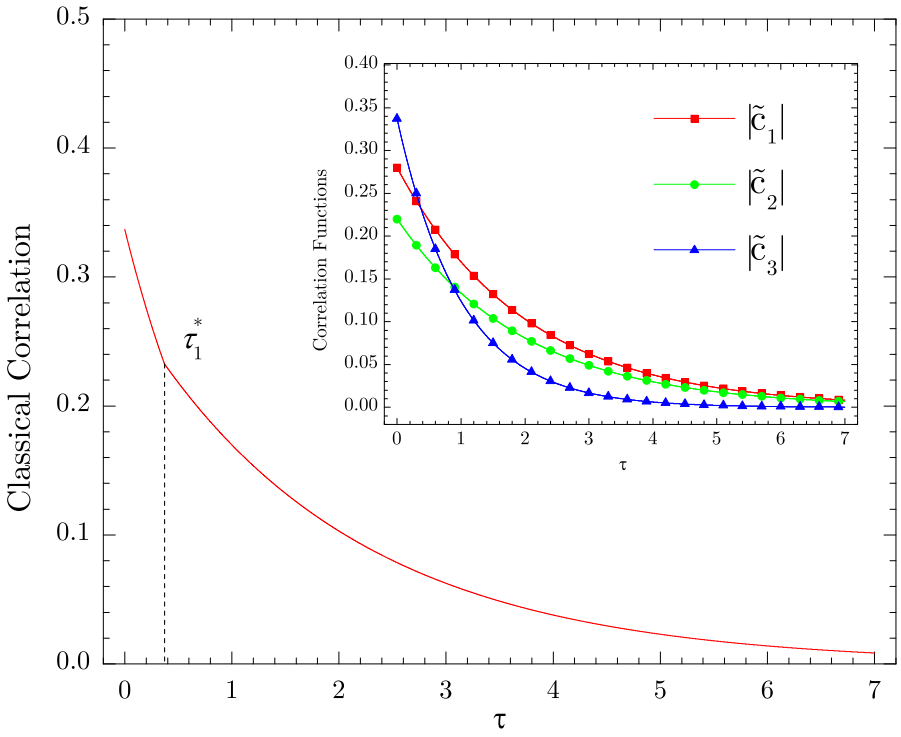

Figure 8 – Classical correlation as a function of $\tau = (\gamma_A + \gamma_B)t$ for a two-qubit system under the GAD channel. The initial state is in the X form, where the values for c_i are selected to show the behavior of the sudden transition, with $c_1 = 0.28$, $c_2 = 0.22$, $c_3 = 0.40$, $c_4 = 0.10$, and $c_5 = 0.60$. A sudden transition in C_G occurs at $\tau_1^* = 0.37$. In the inset, we show the correlation parameters $|\tilde{c}_1|$, $|\tilde{c}_2|$, and $|\tilde{c}_3|$.

5.3.2 Pointer basis for Markovian dynamics

Let us now apply the classical correlation C_G for X states to investigate the emergence of the pointer basis of a quantum apparatus $\mathcal{A}$ subject to decoherence in a Markovian regime. In the measurement-like scenario the pointer basis can be understood as a set of orthonormal states of the apparatus in which correlations with the system state are eventually established (irrespective of the initial states of the apparatus).

The interaction between a quantum measurement apparatus $\mathcal{A}$ and the system of interest $\mathcal{S}$, causes the emergence the classical limit through the process of decoherence. During the measurement the environment E interacts with the measuring apparatus $\mathcal{A}$ collapsing it into a possible set of classical states known as the pointer basis (ZUREK, 2003; ZUREK, 1981; ZUREK, 1982; CORNELIO et al., 2012). Thus, the system-apparatus state ρ_{AS} is decohered [1] into a new classical set of pointer basis with vanishing discord

[1] Decoherence leads to environment-induced superselection which singles out the pointer basis and thus

$D(S|A) = 0$ and which are not altered by decoherence. The pointer basis can be defined as the basis that minimized the discord (to zero). The correlations between these states and the system are preserved, despite the dissipative decoherence process. Hence, decoherence selects the classical pointer states of $\mathcal{A}$, inducing a transition from quantum to classical states of the measurement apparatus.

Therefore, considering the quantum measurement apparatus $\mathcal{A}$ interacting with the system of interest $\mathcal{S}$, with corresponding Hilbert spaces $\mathcal{H}_S$ and $\mathcal{H}_A$ and bases $\{|s_i\rangle\}$ and $\{|A_i\rangle\}$. For the initial state $|\psi_S\rangle$ in $\mathcal{S}$, the composite system $\mathcal{S} + \mathcal{A} + E$ evolves from the initial state $|\psi_S\rangle|A_0\rangle|E_0\rangle$ to $\sum_i c_i|s_i\rangle|A_i\rangle|E_i\rangle$, where $|A_i\rangle$ denotes the orthogonal states of the apparatus, and $|E_i(t)\rangle$ stand for the states of the environment, which are inaccessible to the observer. Hence, the reduced density matrix of the system and the apparatus is given by (ZUREK, 1981; ZUREK, 2003; CORNELIO et al., 2012):

$$\rho_{AS} = \sum_{i,j} c_i c_j^* \langle E_j(t)|E_i(t)\rangle |s_i\rangle\langle s_j|\langle A_j|, \tag{5.26}$$

where the coefficients $\langle E_j(t)|E_i(t)\rangle$ with $i \neq j$ decay rapidly in time. Consequently, the state of $\mathcal{A} + \mathcal{S}$ after decoherence time τ_E is described by

$$\rho_{AS} = \sum_i |c_i|^2 |s_i\rangle|A_i\rangle\langle s_i|\langle A_i|, \tag{5.27}$$

where the states of the bases $\{|s_i\rangle\}$ and $\{|A_i\rangle\}$ are classically correlated. Through of this correlation, an observed can obtain information about $\mathcal{S}$ via measurements on $\mathcal{A}$. Therefore, the environment selects a basis set of classical pointer states $\{|A_i\rangle\}$ of the apparatus and τ_E corresponds to the time for the pointer basis to emerge. The classical correlations between these states and the system are preserved, despite the dissipative decoherence process. Hence, decoherence selects the classical pointer states of $\mathcal{A}$, inducing a transition from quantum to classical states of the measurement apparatus. As a consequence, the information about $\mathcal{S}$ turns out to be accessible to a classical observer through the pointer basis associated with the apparatus. The emergence of the pointer basis occurs for an instant of time τ_E at which the classical correlation between $\mathcal{A}$ and $\mathcal{S}$ becomes constant (CORNELIO et al., 2012; MONTEALEGRE et al., 2013; PAULA et al., 2013a). The classical correlation can be used to characterize the time τ_E when the pointer state emerges, which exactly corresponds to the instant of time at which $C_G(t)$ shows a sudden transition to a constant function.

For our work, we assume that the composite system $\mathcal{AS}$ starts as a X state and the environment only affects the state of the apparatus $\mathcal{A}$ and we focus in an environment E in the form of a PD or GAD channel. We begin by analyzing the case of GAD channel. As we have seen in Fig. 8, the classical correlation for a initial X state under GAD channel

removes quantum excess of correlation responsible for the basis ambiguity.

suffers a sudden change, but continues to decay asymptotically. Consequently, for the GAD channel, there is no emergence of pointer basis at a finite time, since no decay of C_G to a constant function of time is possible [2]. For case of PD channel, the evolution of the classical correlation as function of the dimensionless parameter $\tau = (\gamma_A + \gamma_B)\, t$ is shows in Fig. 9. In this case, the classical correlation decay monotonically until a time τ_E, when a sudden transition occurs and remains constant. For our example in Fig. 9, the emergence of the pointer basis through the behavior of C_G occurs then at $\tau_1^* = 0.92$, i.e., $\tau_E = 0.92\, \tau_D$.

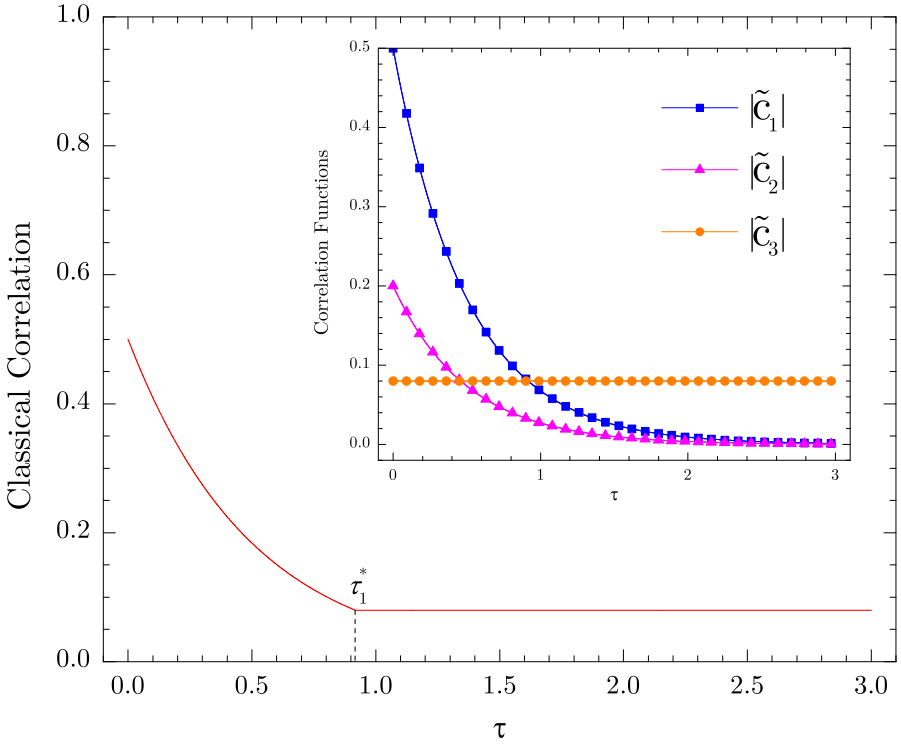

Figure 9 – Classical correlation as a function of $\tau = (\gamma_A + \gamma_B)t$ for a two-qubit system under the PD channel. The initial state is in the X form, where $c_1 = 0.50$, $c_2 = 0.20$, $c_3 = 0.10$, $c_4 = 0.10$, and $c_5 = 0.20$, with these values chosen to illustrate the emergence of the pointer basis. This occurs at $\tau_1^* = 0.92$, i.e., $\tau_E = 0.92\, \tau_D$. In the inset, we detail the evolution of the correlation parameters $|\tilde{c}_1|$, $|\tilde{c}_2|$, and $|\tilde{c}_3|$.

Remarkably, we can analytically determine τ_E for X states. From Table 2, $C_G(t)$

[2] This behavior also can be observed from the decay monotonically of all correlations parameters $\tilde{c}_i$, see Table 2.

gets constant after a sudden transition at finite time given by

$$\tau_E = \tau_D \ln \left[\frac{\tilde{c}_+}{|\tilde{c}_3|} \right]. \tag{5.28}$$

Comparing τ_E with the decoherence time scale τ_D, we can observe that the pointer basis may emerge at a time smaller or larger than τ_D. This generalizes the result obtained in section 4.3.2.1 for Bell-diagonal states (CORNELIO et al., 2012; MONTEALEGRE et al., 2013; PAULA et al., 2013a) .

5.3.3 Non-Markovian dynamics

In the last application, we considered the classical correlations for X states in a non-Markovian open quantum system under the PD channel. The non-Markovian dynamics describes many physical situations, e.g, single flourescent systems hosted in complex environments, biological systems, superconducting qubits, dephasing in atomic and molecular physics, etc.(WOLF et al., 2008; APOLLARO et al., 2011; REBENTROST; CHAKRABORTY; ASPURU-GUZIK, 2009; LIANG, 2010; CHEN et al., 2015; BUDINI, 2006c; WONG; GRUEBELE, 2001). Here, we discuss the class of non-Markovian master equations showed in section 3.2. In this formalism proposed in (BUDINI, 2006b), the system state is written as a sum of auxiliary matrices whose evolution involve Lindblad contributions with local coupling between all them, resembling the structure of a classical rate equation.

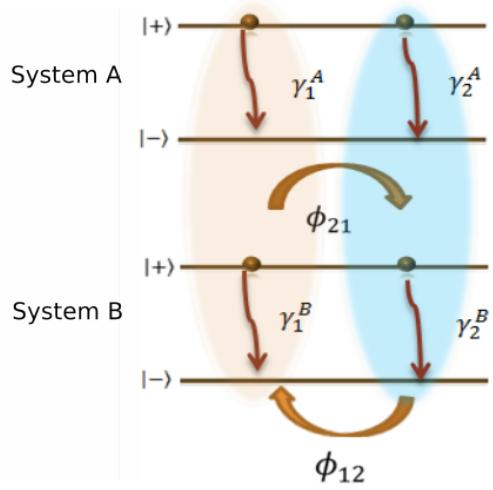

Figure 10 – Two-qubit system A and B interacting with a self-fluctuating environment. We assume an environment with only two subspaces, R $= 1, 2$, which only affects on the decay rates of the system.

In this scenario, we analyze the case of a two-qubit system in the X form interacting with a dispersive reservoir whose action can be written in terms of a dispersive Lindbland rate equation (see section 2.5). Therefore, in this non-Markovian framework, we assumed

a complex reservoir with only two subspaces in Eq. (3.26), namely $R = 1, 2$. The density matrix $\rho_S(t)$ of the system can be written as

$$\rho_S(t) = P_1\rho_1(t) + P_2\rho_2, \tag{5.29}$$

where each auxiliary (unnormalized) operator ρ_R defines the system dynamics given that the reservoir is in the R-configurational bath state, whose statistical weights Eq. (3.27) satisfy $P_1 + P_2 = 1$. It is important to stress that the set of states $\{\rho_R(t)\}$ encodes both the system dynamics and the fluctuations of the environment (BUDINI, 2006b; BREUER, 2007). For our work, we restrict our attention to self-fluctuating environment, this kind of enviromental fluctuations represent situations in which the transitions between the configurational states do not depend on the system state (BUDINI, 2006b; BUDINI, 2008). Thus, the fluctuations between the configurational states are governed by a classical master equation (KAMPEN, 1992) (see Appendix A, Eq. (A.12)). Hence, we model the environment as being characterized by a two-dimensional configurational space only affects the decay rates of the system. A schematic diagram is presented in Fig. 10. The evolution of the auxiliary states follows by itself a Markovian master equation, this is (BUDINI, 2006b; BUDINI, 2008; BUDINI, 2010):

$$\begin{aligned}\frac{d\rho_1(t)}{dt} &= -i[H_1, \rho_1(t)] + \gamma_1^A(\mathcal{L}^A[\rho_1(t)]) + \gamma_1^B(\mathcal{L}^B[\rho_1(t)]) \\ &- \phi_{21}\rho_1(t) + \phi_{12}\rho_2(t),\end{aligned} \tag{5.30}$$

$$\begin{aligned}\frac{d\rho_2(t)}{dt} &= -i[H_2, \rho_2(t)] + \gamma_2^A(\mathcal{L}^A[\rho_2(t)]) + \gamma_2^B(\mathcal{L}^B[\rho_2(t)]) \\ &- \phi_{21}\rho_2(t) + \phi_{12}\rho_1(t),\end{aligned} \tag{5.31}$$

where the structure of the superoperator $\mathcal{L}$ for the PD channel is given by

$$\mathcal{L}^{A,B}[\bullet] = (\sigma_z^{A,B} \bullet \sigma_z^{A,B} - \bullet), \tag{5.32}$$

where σ_z is the z Pauli matrix. The completely CP conditions (BUDINI, 2006b; BUDINI, 2010) imply:

$$\gamma_1^{A,B} \geq 0, \quad \gamma_2^{A,B} \geq 0, \tag{5.33}$$

$$\phi_{12} \geq 0, \quad \phi_{21} \geq 0. \tag{5.34}$$

The first line of Eqs. (5.30) and (5.31) defines the unitary and dissipative dynamics for the two-qubit system, given that the bath is in the configurational state 1 or 2, respectively. The constants $\{\gamma_{1,2}^A, \gamma_{1,2}^B\}$ are the natural decay rates of the system associated with each reservoir state (NIELSEN; CHUANG, 2000). On the other hand, the second

line of Eqs. (5.30) and (5.31) describes transitions between the configurational states of the environment (with rates ϕ_{12} and ϕ_{21}) (BUDINI, 2010). For a matter of simplicity, the decay rates associated with each subsystem will be chosen to be the same, namely, $\gamma_1^A = \gamma_1^B \equiv \gamma_1$ and $\gamma_2^A = \gamma_2^B \equiv \gamma_2$. Moreover, we define the characteristic dimensionless parameters

$$\epsilon = \frac{\gamma_1}{\gamma_1 + \gamma_2}, \quad \epsilon \in [0, 1], \tag{5.35}$$

$$\eta = \frac{\phi_{12}}{\phi_{12} + \phi_{21}}, \quad \eta \in [0, 1], \tag{5.36}$$

$$v = \frac{\phi_{12} + \phi_{21}}{\gamma_1 + \gamma_2}, \quad v \in [0, \infty). \tag{5.37}$$

Similarly as we have done in the Markovian case, we can directly obtain the dynamics of the classical correlations from Eqs. (5.3)-(5.7) and from the definition of C_G in Eq. (5.22). We remark that this non-Markovian PD process preserves the X state form.

Using the new parametrization showed above in terms of ϵ, η and v, we can analyze the system in the limit of either fast or slow environmental fluctuations. The fast limit of environmental fluctuations occurs when the reservoir fluctuations are much faster than the average decay rates of the system, namely, $\{\phi_{R'R}\} \gg \{\gamma_R\}$, which implies $v \gg 1$ from Eq. (5.37). In this limit the system shows Markovian behavior. Conversely, when the bath fluctuations are much slower than the average decay rate, namely, $\{\phi_{R'R}\} \ll \{\gamma_R\}$, the system is in the limit of slow environmental fluctuations. In this case we have $v \ll 1$. In this regime non-Markovian effects become relevant.

Let us now investigate the emergence of the pointer basis for the case of the non-Markovian PD channel, given by Eqs. (5.29)-(5.32). In this scenario, the classical correlation can witness the emergence time τ_E through the sudden transitions in Fig. 11. We also can observe the time of emergence of the pointer basis τ_E is greater for slower environmental fluctuations. Moreover, we observe that for the non-Markovian regime $\{\phi_{R'R}\} \ll \{\gamma_R\}$ the classical correlation displays a bi-exponential decay. Nevertheless, for $\{\phi_{R'R}\} \gg \{\gamma_R\}$. The classical correlation shows a single exponential decay, such as expected for a Markovian behavior. In the next Chapter we will return with the discussion about the characterization of non-Markovianity.

Finally, we focus our attention on the slow configurational transitions. From Fig. 12 is possible to see that τ_E strongly depends on ϵ, i.e., on the ratio of decay rates γ_1 and γ_2. The shortest emergence time occurs for the central value $\epsilon = 0.5$, where decay rates obey $\gamma_1 = \gamma_2$. As we move away from $\epsilon = 0.5$, the emergence of the pointer basis is delayed. In particular, for the limit cases $\epsilon = 0$ or $\epsilon = 1$, the system shows a soft decay, with no sudden transition at finite time.

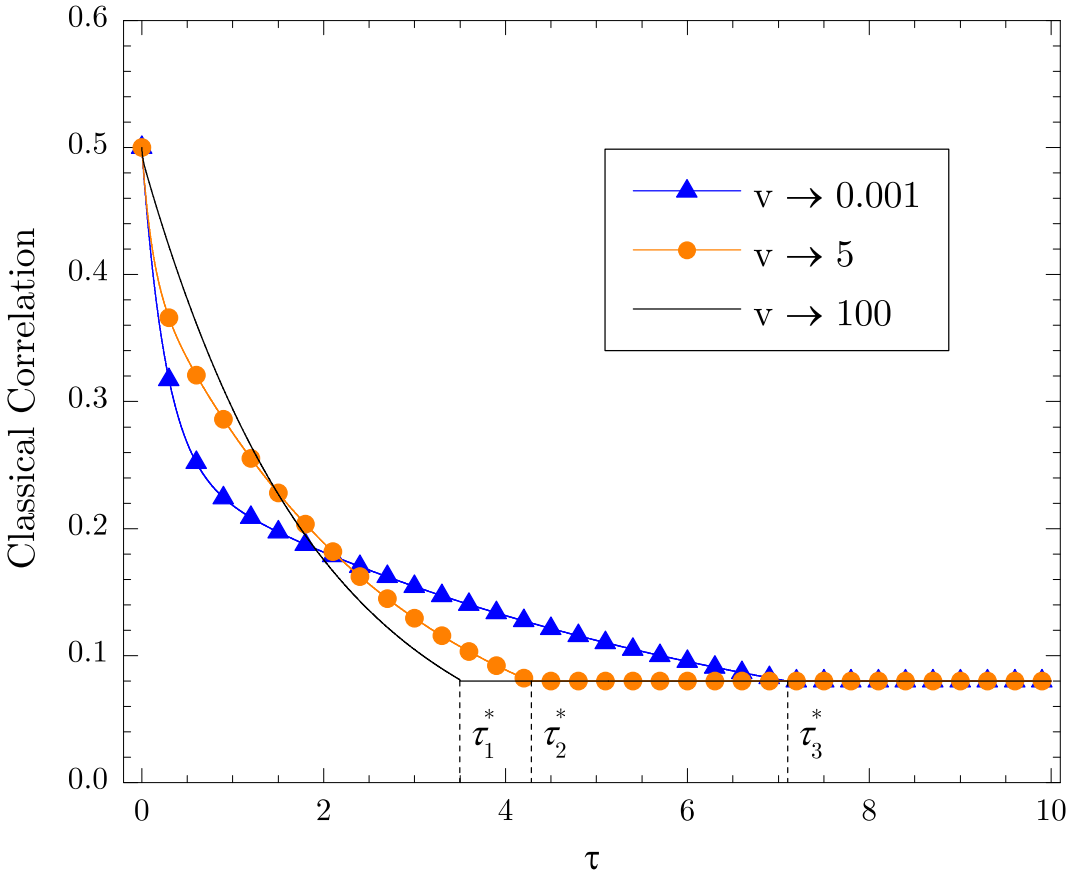

Figure 11 – Classical correlation as a function of $\tau = (\gamma_1 + \gamma_2)t$ for a two-qubit system under the non-Markovian PD channel. The initial state is in the X form, with $c_1 = 0.50$, $c_2 = 0.20$, $c_3 = 0.10$, $c_4 = 0.10$, and $c_5 = 0.20$. We have also taken $\epsilon = 0.92$, $\eta = 0.10$ and probabilities $P_1 = P_2 = \frac{1}{2}$. The emergence times τ_E are associated with $\tau_1^* = 3.5$, $\tau_2^* = 4.3$, and $\tau_3^* = 7.1$. The initial state has been chosen to show the emergence of the pointer basis in the non-Markovian regime.

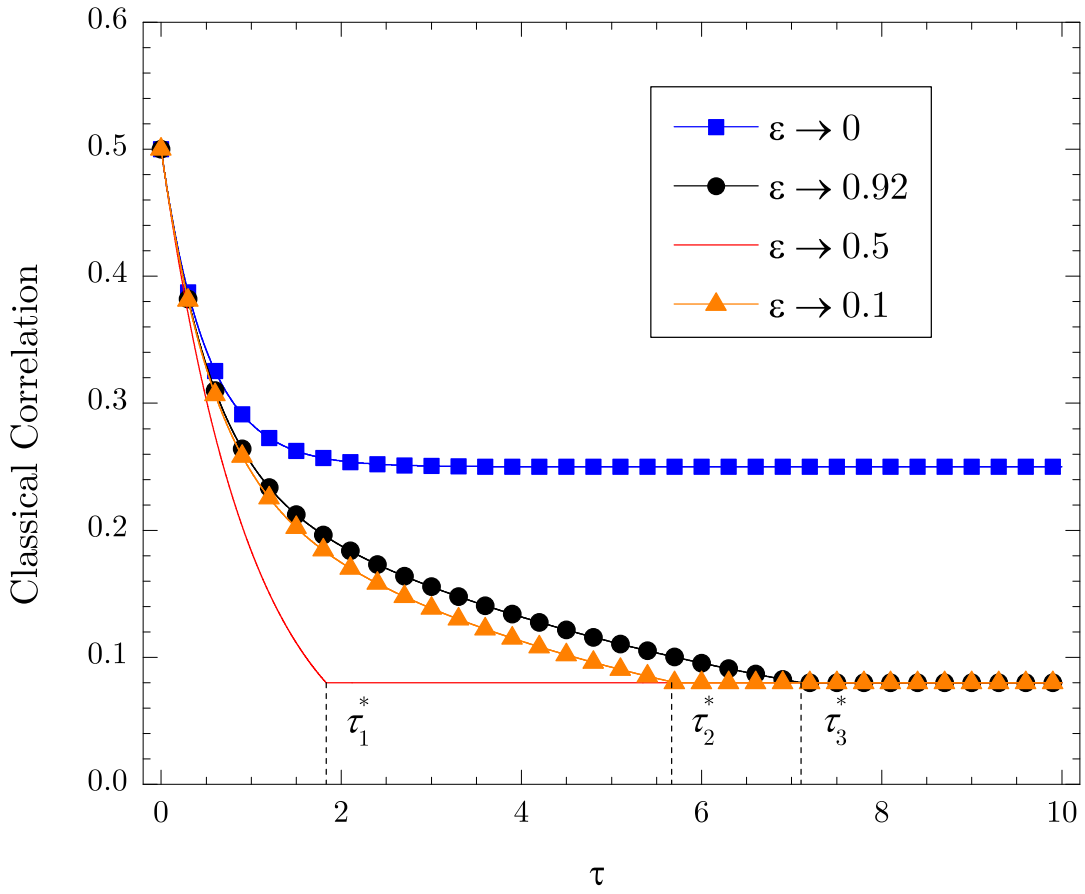

Figure 12 – Classical correlation as a function of $\tau = (\gamma_1 + \gamma_2)t$ for for a two-qubit system under the non-Markovian PD channel in the limit of slow fluctuations, with $v = 0.001$, $\eta = 0.70$ and probabilities $P_1 = P_2 = \frac{1}{2}$. The initial state is in the X form, with $c_1 = 0.50$, $c_2 = 0.20$, $c_3 = 0.10$, $c_4 = 0.10$,and $c_5 = 0.20$. The emergence times τ_E are associated with $\tau_1^* = 1.8$, $\tau_2^* = 5.7$, $\tau_3^* = 7.1$.

6 NON-MARKOVIANITY THROUGH MULTIPARTITE CORRELATION MEASURES

This chapter contains ours results about the characterization of memory effects in non-Markovian systems. In our work we propose a general framework to characterize non-Markovianity through multipartite measures of quantum, classical, and total correlations. We develop our approach for non-Markovianity considering the BLP measure and the program introduced by Rivas, Huelga, and Plenio (RIVAS; HUELGA; PLENIO, 2010), where an ancilla is coupled to a system which interacts with an environment, a review of this approach is showed in section 6.1. Our main result of this chapter is discussed in section 6.2, where we establish sufficient conditions for which generalized measures of multipartite quantum, classical, and total correlations can be used to quantify the degree of non-Markovianity of a local quantum decohering process. In section 6.3 some applications are indicated. We will illustrate our results by considering the dynamical behavior of the trace-distance correlations in multi-qubit systems under local PD and GAD channels.

6.1 Characterizing non-Markovianity

Let us first suppose a quantum process governed by a time-local master equation

$$\frac{d\rho}{dt} = \mathcal{L}_t\,\rho(t), \tag{6.1}$$

where the time-dependent generator $\mathcal{L}_t$ is given by

$$\begin{aligned}\mathcal{L}_t\rho(t) &= -i\left[H(t),\rho(t)\right] + \sum_i \gamma_i(t)\left(A_i(t)\rho(t)A_i^\dagger(t)\right.\\ &\left. -\frac{1}{2}\left\{A_i^\dagger(t)A_i(t),\rho(t)\right\}\right),\end{aligned} \tag{6.2}$$

with $H(t)$ denoting the effective system Hamiltonian, $A_i(t)$ the Lindblad operators, and $\gamma_i(t)$ the relaxation rates. According to section 2.5, the generator $\mathcal{L}_t$ assumes the Lindblad form for each fixed $t \geq 0$, if $\gamma_i(t) \geq 0$. Hence, the dynamics of the density operator can be written as

$$\rho(t) = \Phi_{t,\tau}\rho(\tau),$$

where $\Phi_{t,\tau}$ is a CPTP map, given by

$$\Phi_{t,\tau} = \mathcal{T}_{\leftarrow}\exp\left(\int_\tau^t dt'\mathcal{L}_{t'}\right),$$

with $\mathcal{T}_{\leftarrow}$ denoting the chronological time-ordering operator. The dynamical map $\Phi_{t,\tau}$ then satisfies the divisibility condition

$$\Phi_{t,\tau} = \Phi_{t,r}\Phi_{r,\tau} \ (t \geq r \geq \tau \geq 0),$$

which characterizes the Markovianity of the quantum process. Conversely, for $\gamma_i(t) < 0$, the corresponding dynamical map $\Phi_{t,\tau}$ may not be CPTP for intermediate time intervals and the divisibility property of the overall CPTP dynamics is violated, which characterizes a non-Markovian behavior (BREUER; LAINE; PIILO, 2009; RIVAS; HUELGA; PLENIO, 2010).

If a function $F = F(\rho)$ is monotonically nonincreasing under divisible maps acting on ρ, i.e., $F(\Phi_{t,\tau}\rho(\tau)) \leq F(\rho(\tau))$ when $\Phi_{t,\tau} = \Phi_{t,r}\Phi_{r,\tau}$, then $F(\rho)$ is monotonically nonincreasing with increasing time, namely, $dF(t)/dt \leq 0$. However, this is not always true for a non-Markovian process, where the divisibility of the map is violated, being $dF(t)/dt > 0$ a straightforward non-Markovianity witness (BREUER; LAINE; PIILO, 2009). Therefore, $F(\rho)$ can be employed to point out the breakdown of Markovianity and the degree of non-Markovianity can be defined by

$$N_F(\Phi) = \max_{\rho(0)} \int_{\frac{d}{dt}F(t)>0} \frac{d}{dt}F(t)dt, \tag{6.3}$$

with the maximization performed over all sets of possible initial states, $\rho\,(0)$, and the integration extended over all time intervals for which $dF(\rho)/dt > 0$. Numerically, we can write

$$N_F(\Phi) = \max_{\rho(0)} \sum_i \left[\{F(\tau_{i+1}) - F(\tau_i)\}\right], \tag{6.4}$$

where $\{(\tau_i, \tau_{i+1})\}$ represents the set of all time intervals for which $F(t + \Delta t) > F(t)$. The maximization over $\rho(0)$ is not a trivial task. Nevertheless, it is always possible to find out lower bounds to $N_F(\Phi)$ by optimizing over any class of initial states, which leads to a qualitative assessment of the non-Markovianity of the map Φ (DHAR; BERA; ADESSO, 2015).

6.2 Non-Markovianity through Correlation measures

Let us first come back to the general approach of discord-like measures of quantum, classical, and total correlations of a n-partite system in a state ρ, introduced in section 4.4, which we express by the following definitions:

$$Q(\rho) = \mathcal{K}\left[\rho,\, M^{-}\rho\right], \tag{6.5}$$

$$C(\rho) = \mathcal{K}\left[M^{+}\rho,\, M^{+}\pi_\rho\right], \tag{6.6}$$

$$T(\rho) = \mathcal{K}\left[\rho,\, \pi_\rho\right], \tag{6.7}$$

where $\mathcal{K}[\rho,\sigma]$ denotes a real and positive function that vanishes for $\rho=\sigma$, the operator $\pi_\rho = \mathrm{tr}_{\bar{1}}\rho\otimes\mathrm{tr}_{\bar{2}}\rho...\otimes\mathrm{tr}_{\bar{n}}\rho$ represents the product of the local marginals of ρ, and $M^-\rho$ and $M^+\rho$ are classical states obtained through measurement maps M^- and M^+ that minimize Q and maximize C, respectively. Here, we treat a measurement maps as local n-partite maps

$$M^\pm = M_1^\pm\otimes M_2^\pm\cdots\otimes M_n^\pm, \tag{6.8}$$

where $M_i^\pm\neq\mathbb{I}$ if the ith partition is measured or $M_i^\pm=\mathbb{I}$ if the ith partition is unmeasured. We will consider the measurements $\{M_i^\pm\neq\mathbb{I}\}$ defined by the complete sets of local orthogonal projectors, satisfying:

$$M_i^\pm M_i^\pm\rho = M_i^\pm\rho.$$

As discussed in section 4.4.2, the correlation measures in Eqs. (6.5)-(6.7) are expected to obey the following set of fundamental criteria: (i) product states have no correlations, (ii) all correlations are invariant under local unitary operations, (iii) all correlations are non-negative, (iv) total correlations are nonincreasing under local quantum channels (CPTP maps), (v) classical states have no quantum correlations, and (vi) quantum correlations are nonincreasing under local quantum channels over unmeasured subsystems (PAULA et al., 2014). In order to satisfy the requirements above, we restrict $\mathcal{K}$ to be positive and unitary invariant. An important point to note here is that we requiere $\mathcal{K}$ to be contractible under CPTP maps Φ, i.e., we have (PAULA et al., 2014):

$$\mathcal{K}[\Phi\rho,\Phi\sigma]\leq\mathcal{K}[\rho,\sigma], \tag{6.9}$$

for all states ρ and σ.

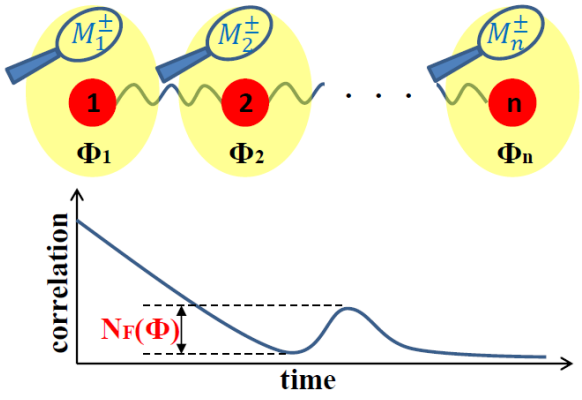

Figure 13 – Illustration of the degree of non-Markovianity $N_F(\Phi)$ via locally measured states.

In the following we will consider multipartite correlated quantum systems, such as illustrated by Fig. 13. Particularly, we will consider local dynamical maps $\Phi=\Phi_1\otimes$

$\Phi_2 \cdots \otimes \Phi_n$. In this scenario, we can show that a non-monotonic behavior of the correlations $Q(\rho)$, $C(\rho)$, and $T(\rho)$ as a function of time may provide a direct measure of the degree of non-Markovianity $N_F(\Phi)$, with $F = Q, C$, or T. This result is established by our next theorem.

Theorem 2. *Consider a quantum evolution driven by a local dynamical map* $\Phi = \bigotimes_{i=1}^{n} \Phi_i$. *Then, assuming that* $\mathcal{K}$ *is contractible under CPTP maps, it follows that:*

1. $N_T(\Phi)$ *is a measure of non-Markovianity;*
2. $N_Q(\Phi)$ *and* $N_C(\Phi)$ *are measures of non-Markovianity for local measurements* $M^{\pm} = \bigotimes_{i=1}^{n} M_i^{\pm}$ *such that* $M_i^{\pm} = \mathbb{I}$ *when* $\Phi_i \neq \mathbb{I}$.

Proof. Our proof starts by observing that $\Phi = \bigotimes_{i=1}^{n} \Phi_i$. Then, by adopting $t \geq \tau \geq 0$, we can write

$$\rho(t) = \Phi_{t,\tau}\rho(\tau) = \bigotimes_i \Phi_{i\,t,\tau}\,\rho(\tau).$$

Moreover, for local CPTP maps, we have the relation

$$\pi_{\Phi_{t,\tau}\rho(\tau)} = \Phi_{t,\tau}\pi_{\rho(\tau)}.$$

- Let us first examine $N_T(\Phi)$. A generalized total correlation measure can be written as

  $$T(t) = \mathcal{K}\left[\rho(t),\, \pi_{\rho(t)}\right] = \mathcal{K}\left[\Phi_{t,\tau}\rho(\tau),\, \Phi_{t,\tau}\pi_{\rho(\tau)}\right].$$

  Imposing that $\mathcal{K}$ is contractible under CPTP maps Eq. (6.9), we have

  $$\mathcal{K}\left[\Phi_{t,\tau}\rho(\tau),\, \Phi_{t,\tau}\pi_{\rho(\tau)}\right] \leq \mathcal{K}\left[\rho(\tau),\, \pi_{\rho(\tau)}\right] = T(\tau).$$

  Thus, we obtain $T(t) \leq T(\tau)$.

- Let us now analyse $N_Q(\Phi)$. A generalized quantum correlation measure can be written as

  $$Q(t) = \mathcal{K}\left[\rho(t), M_t^{-}\rho(t)\right],$$

  where we are considering $M_t^{-} = \bigotimes_i M_{i\,t}^{-}$ with $M_i^{-} = \mathbb{I}$ when $\Phi_i \neq \mathbb{I}$ (consequently, $M^{-}\Phi = \Phi M^{-}$). As M_τ^{-} does not necessarily minimize $Q(t)$, we can write

  $$Q(t) \leq \mathcal{K}\left[\rho(t), M_\tau^{-}\rho(t)\right] = \mathcal{K}\left[\Phi_{t,\tau}\rho(\tau), \Phi_{t,\tau}M_\tau^{-}\rho(\tau)\right].$$

The proof is completed by taking $\Phi_{it,\tau}$ as a CPTP map and imposing that $\mathcal{K}$ is contractible under CPTP maps. Therefore,

$$\mathcal{K}\left[\Phi_{t,\tau}\rho(\tau), \Phi_{t,\tau}M_{\tau}^{-}\rho(\tau)\right] \leq \mathcal{K}\left[\rho(\tau), M_{\tau}^{-}\rho(\tau)\right] = Q(\tau).$$

Clearly $Q(t) \leq Q(\tau)$.

- Finally, let us consider $N_C(\Phi)$. A generalized classical correlation measure can be written as

$$C(t) = \mathcal{K}\left[M_t^{+}\rho(t), M_t^{+}\pi_{\rho(t)}\right],$$

where we are considering $M_t^{+} = \bigotimes_i M_{it}^{+}$ with $M_i^{+} = \mathbb{I}$ when $\Phi_i \neq \mathbb{I}$ (consequently, $M^{+}\Phi = \Phi M^{+}$). Following the same of line thought above, we need to impose that K is contractible under CPTP maps. Hence:

$$C(t) = \mathcal{K}\left[\Phi_{t,\tau}M_t^{+}\rho(\tau), \Phi_{t,\tau}M_t^{+}\pi_{\rho(\tau)}\right] \leq \mathcal{K}\left[M_t^{+}\rho(\tau), M_t^{+}\pi_{\rho(\tau)}\right].$$

As M_t^{+} does not necessarily maximize $C(\tau)$, then

$$\mathcal{K}\left[M_t^{+}\rho(\tau), M_t^{+}\pi_{\rho(\tau)}\right] \leq \mathcal{K}\left[M_\tau^{+}\rho(\tau), M_\tau^{+}\pi_{\rho(\tau)}\right] = C(\tau).$$

Hence, $C(t) \leq C(\tau)$. This completes the proof.

□

Our theorem provides a criterion for measuring non-Markovianity by quantum correlations in a general approach. This theorem ensures $N_Q(\Phi)$ and $N_C(\Phi)$ as measures of non-Markovianity. By assuming that the subsystems under decoherence are unmeasured, i.e., $M_i^{\pm} = \mathbb{I}$ when $\Phi_i \neq \mathbb{I}$. The measurements are then performed over ancillary states that are effectively free of decoherence. One can see from Eq. (6.7) that this requirement is unnecessary for $N_T(\Phi)$, since it is a measurement-independent quantifier.

An effective isolation of an ancilla to probe non-Markovianity has been experimentally achieved in several scenarios (see for example the references (XU et al., 2013; BERNARDES et al., 2015)). In general terms, it can be approximately assumed when the relaxation times of the ancillary subsystem are much larger than those of the principal subsystem. In the same way, how we will show in Eq. (6.11) for a bipartite example in section 6.3.1, it also happens when the multi-local dynamical map can be written as an effective transformation where only part of the subsystems undergoes decoherence. In the following section some applications are indicated.

6.3 Applications

In the following we will illustrate some applications of our theorem by looking at three types of non-Markovian dynamics: the PD channel, the GAD via Lindblad rate equation and multi-qubit systems under local PD.

6.3.1 Two-qubit state under PD noise via a time-local master equation

Let us start by considering a local PD master equation. The Hamiltonian in Eq. (6.2) for this case can be written as

$$H = -\tfrac{1}{2}\,(\sigma_z^1 + \sigma_z^2),$$

where σ_z is the standard z Pauli matrix and the A_j operators in Eq. (6.2) are given by $A_j = \sigma_z^j$ with $j = 1, 2$. Solutions of this local PD master equation can be obtained via several routes, and we find the operator sum representation convenient for our purpose. Thus, we consider the quantum dynamical map of the form $\Phi = \Phi_1 \otimes \Phi_2$, where[1]:

$$\Phi_j\rho_j = \frac{1 + f_j(t)}{2}\rho_j \;+\; \frac{1 - f_j(t)}{2}\sigma_z^j\rho_j\sigma_z^j, \tag{6.10}$$

with $f_j(t) = \exp\left[-2\int_0^t \gamma_j(\tau)d\tau\right]$. For simplicity, the time-dependent decoherence rates associated with each subsystem will be chosen to be the same, namely,

$$\gamma_1(t) = \gamma_2(t) \text{ such that } f_1(t) = f_2(t).$$

One finds that the map Φ_j preserves the X form of the any initial X state given by Eq. (5.2). That is

$$\{c_1(t), c_2(t), c_3(t), c_4(t), c_5(t)\} = \{c_1 f_k(t), c_2 f_k(t), c_3, c_4, c_5\}.$$

Moreover, we can write

$$\Phi = \Phi_1 \otimes \Phi_2 = \Phi^{eff} \otimes \mathbb{I} = \mathbb{I} \otimes \Phi^{eff}, \tag{6.11}$$

where Φ^{eff} is an effective PD channel with

$$f(t) = f_1(t)f_2(t) = \exp\left[-2\int_0^t \gamma(\tau)d\tau\right],$$

with $\gamma(\tau) = \gamma_1(\tau) + \gamma_2(\tau) = 2\gamma_1(t)$, such that

[1] Here the operator-sum representation for each qubit is given by $\Phi\rho_j = \sum_{i=1}^{2} K_i^j(t)\rho K_i^{j\dagger}(t)$ with the time-dependent Kraus operators: $K_1^j(t) = \sqrt{\frac{1+f_j(t)}{2}}\mathbb{I}$ and $K_2^j(t) = \sqrt{\frac{1-f_j(t)}{2}}\sigma_z$

$$\{c_1(t), c_2(t), c_3(t), c_4(t), c_5(t)\} = \{c_1 f(t), c_2 f(t), c_3, c_4, c_5\}.$$

In this way, we can use our Theorem 1 to characterize the non-Markovianity of an effective PD channel by quantum, classical or total correlations defined in Eqs. (5.14, 5.22, 5.23) respectively. Note that $\gamma(t)$ needs not be positive. If $\gamma(t) \geq 0$ for all $t \geq 0$, then the channel is in a Markovian regime. Therefore, we have $df(t)/dt \leq 0$, and as a consequence we have that $dQ(t)/dt \leq 0$, $dC(t)/dt \leq 0$, and $dT(t)/dt \leq 0$. It follows immediately that

$$N_Q = N_C = N_T = 0.$$

In order to quantify the non-Markovianity of the dynamical map $\Phi = \Phi_1 \otimes \Phi_2$, we will first consider the classical correlation (by replacing $C(t)$ for $F(t)$ in Eq. (6.3)). The condition $dC(t)/dt > 0$ occurs if and only if

$$C(t) > |c_3(0) - c_4(0)c_5(0)| \text{ and } \gamma(t) < 0.$$

Therefore,

$$\tfrac{dC(t)}{dt} = -2\max\{|c_1(0)|, |c_2(0)|\}\gamma(t)f(t),$$

and the maximization in Eq. (6.3) is achieved when $c_3(0) - c_4(0)c_5(0) = 0$ and $\max\{|c_1(0)|, |c_2(0)|\} = 1$. Thus, we find

$$N_C(\Phi) = -2\int_{\gamma(t)<0} \gamma(t)f(t)dt, \tag{6.12}$$

where we taking $\rho(0)$ as a maximally entangled state. Remarkably, we obtain the same estimation of non-Markovianity degree via total correlation $T(t)$ or quantum correlation $Q(t)$.

It is interesting to highlight that for the particular case of Bell states, this is $|c_1| = |c_2| = |c_3| = 1$ and $c_4 = c_5 = 0$, the total correlation and quantum correlations are given by

$$T(t) = \max\{1, f(t) + \tfrac{1}{2}\} \text{ and } Q(t) = f(t).$$

As $df(t)/dt = -2\gamma(t)f(t)$ and $f(t) > 0$, we conclude that $dT(t)/dt > 0$ or $dQ(t)/dt > 0$ is equivalent to $\gamma(t) < 0$. Under this condition, $f(t) > 1$ and $dT(t)/dt = dQ(t)/dt = -2\gamma(t)f(t)$, which leads to

$$N_T(\Phi) = N_Q(\Phi) = N_C(\Phi). \tag{6.13}$$

The degree of non-Markovianity found here via trace-distance correlation measures is then consistent with previous results for measures of non-Markovianity (BREUER; LAINE;

PIILO, 2009; DHAR; BERA; ADESSO, 2015; VACCHINI et al., 2011; LUO; FU; SONG, 2012).

6.3.2 Two-qubit state under GAD via Lindblad rate equation

Let us now illustrate the signature of non-Markovianity for two-qubits under local GAD, which will be described in the open-systems framework developed by Budini (BUDINI, 2006b) based on a Lindblad rate equation (see for more details the non-Markovianity Chapter 3, section 3.2). In this scenario, we model the environment as being characterized by a two-dimensional configurational space ($R_{max} = 2$) which only affects the decay rates of the system. The reduced density operator of the system Eq. (6.1) is obtained by (BUDINI, 2006b; BUDINI, 2005):

$$\rho_S(t) = \sum_{R=1}^{R_{max}} P_R \rho_R(t), \tag{6.14}$$

where the probability that the environment is in a given state at time t is given by Eq. (3.27). Each state follows by itself a Lindblad rate equation

$$\begin{aligned} \frac{d\rho_1(t)}{dt} &= -i[H_1, \rho_1(t)] + \bar{\gamma}_1^A \mathcal{L}^A \rho_1(t) + \bar{\gamma}_1^B \mathcal{L}^B \rho_1(t) \\ &- \phi_{21}\rho_1(t) + \phi_{12}\rho_2(t), \end{aligned} \tag{6.15}$$

$$\begin{aligned} \frac{d\rho_2(t)}{dt} &= -i[H_2, \rho_2(t)] + \bar{\gamma}_2^A \mathcal{L}^A \rho_2(t) + \bar{\gamma}_2^B \mathcal{L}^B \rho_2(t) \\ &- \phi_{21}\rho_2(t) + \phi_{12}\rho_1(t), \end{aligned} \tag{6.16}$$

where the structure of the superoperator $\mathcal{L}$ for the GAD channel is given by

$$\mathcal{L}^{A,B}\bullet = \left(\frac{\sigma^{\dagger A,B}\sigma^{A,B}\bullet}{2} + \frac{\bullet\,\sigma^{\dagger A,B}\sigma^{A,B}}{2} - \sigma^{\dagger A,B}\bullet\,\sigma^{A,B}\right). \tag{6.17}$$

In the above equations both $\rho_1(t)$ and $\rho_2(t)$ follow a Lindbland type of evolution equation of the form (2.45) induced by the coupling with the corresponding subreservoir. Therefore, we can observe that the first lines of Eqs. (6.15) and (6.16) define the unitary and dissipative dynamics for the two-qubit system, given that the bath is in the configurational state 1 or 2, respectively. The constants $\{\bar{\gamma}_{1,2}^A, \bar{\gamma}_{1,2}^B\}$ are the natural decay rates of the system associated with each reservoir state. The positivity of the density matrix will be ensured as long as these decoherence coefficients obey $\bar{\gamma}_{1,2}^{A,B} \geq 0$ (BUDINI, 2006b; BUDINI, 2010).

Moreover, the transitions between the configurational states ρ_1 and ρ_2 is described by the second line of Eqs. (6.15) and (6.16), with rates of transitions ϕ_{12} and ϕ_{21}.

For a matter of simplicity, the decay rates associated with each subsystem will be chosen to be the same, namely, $\bar{\gamma}_1^A = \bar{\gamma}_1^B \equiv \bar{\gamma}_1$ and $\bar{\gamma}_2^A = \bar{\gamma}_2^B \equiv \bar{\gamma}_2$. Moreover, we define the characteristic dimensionless parameters

$$\epsilon = \frac{\bar{\gamma}_1}{\bar{\gamma}_1 + \bar{\gamma}_2}, \quad \eta = \frac{\phi_{12}}{\phi_{12} + \phi_{21}} \quad v = \frac{\phi_{12} + \phi_{21}}{\bar{\gamma}_1 + \bar{\gamma}_2}, \tag{6.18}$$

where $\epsilon, \eta \in [0,1]$ and $v \in [0,\infty)$. We will analyze the system in the limit of either fast or slow environmental fluctuations. The fast limit of environmental fluctuations occurs when the reservoir fluctuations are much faster than the average decay rates of the system, namely, $\{\phi_{R'R}\} \gg \{\bar{\gamma}_R\}$ ($v \gg 1$), which implies that the system exhibits Markovian behavior. On the other hand, when the bath fluctuations are much slower than the average decay rate, namely, $\{\phi_{R'R}\} \ll \{\bar{\gamma}_R\}$ ($v \ll 1$), the system is in the limit of slow environmental fluctuations. The signatures of non-Markovianity will be provided by the total correlation, which has the advantages of avoiding both extremization procedures and further requirements over the dynamical map. The non-Markovian behavior can then be witnessed in Fig. 14, which shows the temporal evolution of the total correlation for several values of v, where we have taken $\epsilon = 0.92$, $\eta = 0,5$, and an initial X state described by $c_1 = 0.20, c_2 = -0.20, c_3 = 0.60, c_4 = 0.50$ and $c_5 = 0.70$.

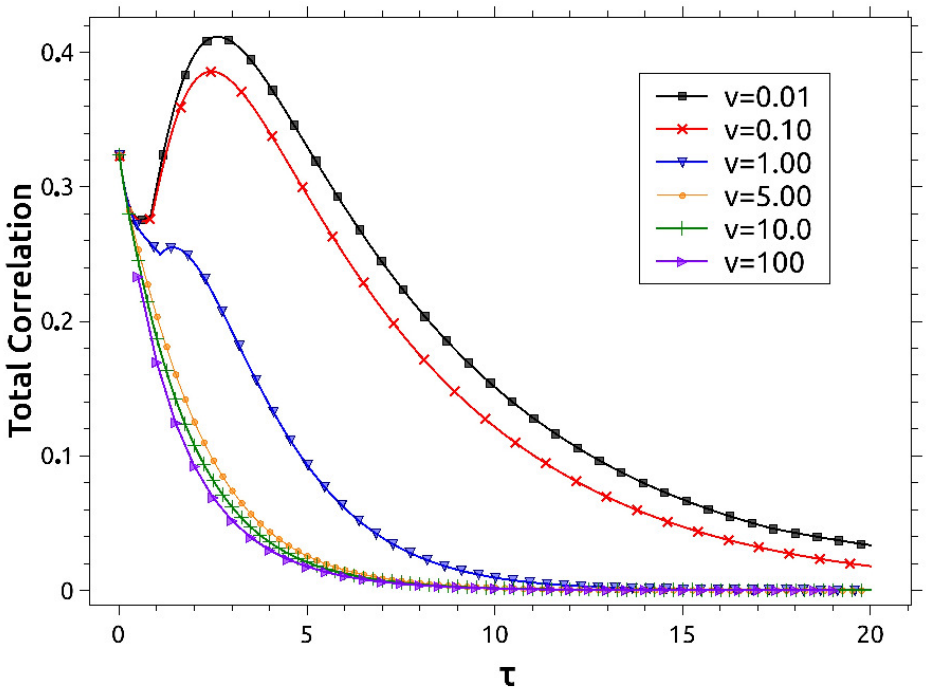

Figure 14 – Total Correlation as a function of $\tau = (\bar{\gamma}_A + \bar{\gamma}_B)t$ for a two-qubit system under non-Markovian GAD channel. The initial state is in the X form, with $c_1 = 0.20$, $c_2 = -0.20$, $c_3 = 0.60$, $c_4 = 0.50$, and $c_5 = 0.70$. We have also taken $P_1 = P_2 = \frac{1}{2}$, $\epsilon = 0.92$ and $\eta = 0.5$. The non-monotonic behavior gets pronounced as we decrease v from the Markovian regime ($v \gg 1$) towards the non-Markovian regime.

For the fast limit, the decay of the total correlation is a monotonically decreasing function, which corresponds to a Markovian evolution. Otherwise, when the system is subject to a non-Markovian evolution (slow limit), the total correlation shows a non-monotonic evolution, which gets more pronounced as we decrease v. The degree of non-Markovianity $N_T(v)$ can be rigorously obtained from Eq. (6.4) by a maximization over all

initial states. On the other hand, a lower bound for $N_T(v)$ can be directly obtained from Fig. 14 through the height of the non-monotonic sector as a function of v.

6.3.3 Multipartite entangled state under local PD

Finally, we briefly discuss the multipartite total correlation as a non-Markovian quantifier. Let us consider an n-partite system initially in the the Greenberger- Horne-Zeilinger (GHZ) state (GREENBERGER; HORNE; ZEILINGER, 1989), i.e., a maximally entangled state of the form

$$\rho = \frac{1}{2}\left(|0\rangle^n\langle 0|^n + |0\rangle^n\langle 1|^n + |1\rangle^n\langle 0|^n + |1\rangle^n\langle 1|^n\right), \tag{6.19}$$

where $|k\rangle^n = |k\rangle_1 \otimes |k\rangle_2 \otimes \cdots \otimes |k\rangle_n$ $(k = 0, 1)$. By applying a dynamical map $\Phi = \Phi_1 \otimes \Phi_2 \otimes \cdots \otimes \Phi_n$ over the GHZ state and choosing Φ_i as a local PD channel such as in Eq. (6.10), we get

$$\rho(t) = \frac{1}{2}\left(|0\rangle^n\langle 0|^n + f(t)\,|0\rangle^n\langle 1|^n + f(t)\,|1\rangle^n\langle 0|^n + |1\rangle^n\langle 1|^n\right), \tag{6.20}$$

where $f(t) = \exp\left[-2\int_0^t \gamma(\tau)d\tau\right]$ with $\gamma(t) = \sum_{i=1}^n \gamma_i(t)$ denoting the sum of the time-dependent decoherence rates.

We will consider the multipartite total correlation as the non-Markovianity quantifier and use the GHZ state to provide a lower bound for $N_T(\Phi)$. For this purpose, we find that the product of the local marginals of $\rho(t)$ is given by $\pi_{\rho(t)} = \mathbb{I}/2^n$ and the eingenvalues of the operator $\rho(t) - \pi_{\rho(t)}$ are

$$\begin{aligned} \lambda_i &= -2^{-n} \quad (1 \le i \le 2^n - 2), \\ \lambda_{2^n-1} &= \frac{1}{2}\left(1 - 2^{1-n} - f\right), \\ \lambda_{2^n} &= \frac{1}{2}\left(1 - 2^{1-n} + f\right). \end{aligned} \tag{6.21}$$

Therefore, using the total correlation based on the trace-norm given by Eq. (5.13) we find

$$T(t) = \textstyle\sum_i^{2^n} |\lambda_i| = 1 - 2^{1-n} + \max\{1 - 2^{1-n}, f(t)\}.$$

Note that $dT/dt = 0$ for $f(t) \le 1 - 2^{1-n}$, then the measure $N_T = 0$. Furthermore, we find that $dT/dt = -2\gamma(t)f(t)$ for $f(t) > 1 - 2^{1-n}$. It is straightforward to show that $dT/dt > 0$ for $\gamma(t) < 0$. Indeed $f(t) > 1$ (consequently, $f(t) > 1 - 2^{n-1}$) when $\gamma(t) < 0$. Hence, a generalization of our result in Eq. (6.12) is given by:

$$N_T(\Phi) = \int_{\frac{d}{dt}T(t)>0} \frac{d}{dt}T(t)dt = -2\int_{\gamma(t)<0} \gamma(t)f(t)dt. \tag{6.22}$$

7 QUANTUM COHERENCE

Quantum coherence is a fundamental feature of quantum mechanics and is an important physical resource in quantum information (STRELTSOV; ADESSO; PLENIO, 2017; HU et al., 2017; BAUMGRATZ; CRAMER; PLENIO, 2014). It is closely related to almost all the fascinating quantum phenomena from the superposition principle to quantum correlations. The coherent superposition of states, in combination with the quantization of observables, represents one of the most fundamental characteristics that marks the departure of quantum mechanics from the classical realm (STRELTSOV; ADESSO; PLENIO, 2017). Quantum optical methods provide an important set of tools for the manipulation of coherence, and indeed, at its basis lies the formulation of the quantum theory of coherence (GLAUBER, 1963; SUDARSHAN, 1963; STRELTSOV; ADESSO; PLENIO, 2017). While quantum correlations characterize the quantum features of a system with at least two parties, quantum coherence already for a single system (BAUMGRATZ; CRAMER; PLENIO, 2014; STRELTSOV; ADESSO; PLENIO, 2017). As we already mentioned in the introductory Chapter and the quantum correlations Chapter 4, studies over the past decade have provided important information on the geometric characterization of quantum correlations in a bipartite system with the corresponding measures being defined based on various (pseudo) distance measures of two-states (MODI et al., 2011).

Inspired by the recent developments about the quantitative characterization of coherence (BAUMGRATZ; CRAMER; PLENIO, 2014; STRELTSOV; ADESSO; PLENIO, 2017), our purpose in this Chapter is the study of quantum coherence for one-qubit systems. This chapter is organized as follows: In section 7.1 we describe the formalism of quantum coherece. In section 7.2, we analyze in detail the connections of quantum coherence with quantum correlations measures. In section 7.3 we focus on the trace-norm coherence. In section 7.4, we consider dynamics under decohering process. Finally, in section 7.5 we present our original results about the quantification of non-Markovinity by quantum coherence, and give an example with analytic results.

7.1 Quantum coherence measures

Let us begin our discussion defining the characteristics of incoherent states and the notion of incoherent operations. Coherence is naturally a basis dependent concept, for this reason we first need an orthonormal reference basis [1] $\{|k\rangle\}_{k=1,\dots d}$ of the d-dimensional

[1] The reference basis may be dictated by the physics of the problem under investigation or by a task for which coherence is required (STRELTSOV; ADESSO; PLENIO, 2017).

Hilbert space $\mathcal{H}$. The density matrices that are diagonal in this specific basis are called incoherent and form the set of incoherent states $\mathcal{I} \subset \mathcal{H}$. Therefore, all density operators $\delta \in \mathcal{I}$ are of the form (BAUMGRATZ; CRAMER; PLENIO, 2014):

$$\delta = \sum_{k=1}^{d} c_k |k\rangle\langle k|. \tag{7.1}$$

On the other hand, the set of incoherent physical operations is characterized by a set of Krauss operators $\{K_i\}$ satisfying $\sum_i K_i^\dagger K_i = \mathbb{I}$ and fulfilling $K_i \mathcal{I} K_i^\dagger \subset \mathcal{I}$ for all i. There are two classes of incoherent operations (BAUMGRATZ; CRAMER; PLENIO, 2014):

A) The incoherent completely positive and trace-preserving (ICPTP) operations, which act as $\Phi_{\text{ICPTP}}\rho = \sum_i K_i \rho K_i^\dagger$, where the Kraus operators K_i are all of the same dimension and satisfy $K_i \mathcal{I} K_i^\dagger / p_i \subset \mathcal{I}$ for arbitrary $\delta \in \mathcal{I}$, with $p_i = \text{tr}(K_i \rho K_i^\dagger)$ being the probability of obtaining the result i.

B) The incoherent operations with subselection based on the output measurement results being permitted. They also require $K_i \mathcal{I} K_i^\dagger / p_i \subset \mathcal{I}$ to be satisfied for all $\delta \in \mathcal{I}$. But now the dimension of K_i may be different, that is, different K_i may correspond to different output spaces.

The d-dimensional maximally coherent state is defined by (BAUMGRATZ; CRAMER; PLENIO, 2014):

$$|\Psi_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle, \tag{7.2}$$

for which any ρ in the same Hilbert space can be generated with certainty by merely incoherent operations Φ_{ICPTP} on it. It is important to note that Eq. (7.2) is independent of a specific measure for coherence and serves as a unit for defining coherence measures.

Let us now to define the conditions for a proper coherence measure (STRELTSOV; ADESSO; PLENIO, 2017; BAUMGRATZ; CRAMER; PLENIO, 2014):

(C1) Nonnegativity, i.e.,$C(\rho) \geq 0$, and $C(\delta) = 0$ iff $\delta \in \mathcal{I}$.

(C2a) Monotonicity: C does increase under the action of ICPTP map, $C(\rho) \geq C(\Phi_{\text{ICPTP}}\rho)$, for all Φ_{ICPTP}.

(C2b) Strong Monotonicity: C does not increase on average under selective incoherent operations, $C(\rho) \geq \sum_i p_i C(\sigma_i)$ for all incoherent Kraus operators $\{K_i\}$ with probabilities $p_i = \text{tr}[K_i \rho K_i^\dagger]$ and postmeasurement states $\sigma_i = K_i \rho K_i^\dagger / p_i$. This condition quantifies the intuition that coherence should not increase under incoherent measurements even if one has access to the individual measurements outcomes.

(C3) Nonincreasing under mixing of quantum states, i.e., $\sum_i p_i C(\rho_i) \geq C(\sum_i p_i \sigma_i)$ for any set of states $\{\rho_i\}$ and any $p_i \geq 0$ with $\sum_i p_i = 1$.

Coherence measures that satisfy conditions (C2b) and (C3) imply condition (C2a). This gives:

$$C(\Phi_{ICPTP}\rho) = C(\sum_i p_i \rho_i) \overset{(C3)}{\leq} \sum_i p_i C(\rho_i)) \overset{(C2b)}{\leq} C(\rho). \tag{7.3}$$

A quantity C which fulfills conditions (C1)-(C3) is called coherence monotone. Some examples of a quantifier of coherence that satisfies the conditions (C1)-(C3) are the distillable coherence and coherence cost (YUAN et al., 2015; WINTER; YANG, 2016), relative entropy of coherence (BAUMGRATZ; CRAMER; PLENIO, 2014), geometric coherence (STRELTSOV et al., 2015), coherence monotones from entanglement (STRELTSOV et al., 2015), robustness of coherence (NAPOLI et al., 2016), etc.

Analogously to the quantum correlation measures of entanglement and quantum discord (see section 4.2), a general distance-based coherence quantifier is defined as (BAUMGRATZ; CRAMER; PLENIO, 2014) :

$$C_{\mathcal{D}}(\rho) = \min_{\delta \in \mathcal{I}} \mathcal{D}(\rho, \delta), \tag{7.4}$$

where $\mathcal{D}(\rho, \delta)$ is the distance (or pseudo distance) between ρ and δ with $\delta \in \mathcal{I}$.

7.2 Connections with quantum correlations measurements

Since coherence is a basis-dependent concept, a unitary operation will in general change the amount of coherence in a given state. However, for several situations, it is convenient to remove the basis dependence by optimizing over local unitaries. Using the general distance-based coherence given by Eq. (7.4), and considering that the nearest incoherent state is given by $\delta = \rho_{diag}$, where ρ_{diag} corresponds to the diagonal part of ρ in a given basis, we can define the following distance-based coherence for $N \geq 2$:

$$C_d(\rho) = \min_{\delta \in \mathcal{I}} \mathcal{D}(\rho, \rho_{diag}), \tag{7.5}$$

where the norm $\mathcal{D}$ is invariant under local unitary operations.

Therefore, using the definition given by Eq. (7.5), the minimum amount of coherence attainable by varying the reference basis is defined by (YAO et al., 2015):

$$C_{\min}(\rho) = \min_{U} C_d(U \rho U^\dagger), \tag{7.6}$$

where C is a valid coherence measure and $U = U_1 \otimes U_2 \otimes .... \otimes U_n$. Following the same line of thought, we can also define a basis-free measure of coherence by the maximization over

all local unitary transformations (YU; YANG; GUO, 2016; HU et al., 2017; STRELTSOV et al., 2016):

$$C_{\max}(\rho) = \max_{U} C_d(U\rho U^\dagger). \tag{7.7}$$

The next theorem tells us that $C_{\min}(\rho)$ is equivalent to the quantum discord.

Theorem 3. *$C_{min}(\rho)$ is a measure of quantum discord $\mathcal{Q}(\rho)$.*

Proof. Let us consider a measure of coherence given by Eq. (7.5) where the nearest incoherent state is given by $\delta = \rho_{diag}$, due that the norm $\mathcal{D}$ is invariant under unitary operations, i.e., $\mathcal{D}(U\rho U^\dagger, U\sigma U^\dagger) = \mathcal{D}(\rho, \sigma)$ and ρ_{diag} stands for the diagonal part of ρ in a basis $\{|k\rangle\}$. We have:

$$\begin{aligned} C_{min}(\rho) &\equiv \min_{U} \mathcal{D}(U\rho U^\dagger, (U\rho U^\dagger)_{diag}), \\ &= \min_{U} \mathcal{D}(UU^\dagger \rho U^\dagger U, U^\dagger (U\rho U^\dagger)_{diag} U), \\ &= \min_{U} \mathcal{D}(\rho, U^\dagger (U\rho U^\dagger)_{diag} U). \end{aligned} \tag{7.8}$$

Since $\rho_{diag} = \Pi_k(\rho) = \sum_k |k\rangle\langle k|\rho|k\rangle\langle k|$, it follows that:

$$U^\dagger (U\rho U^\dagger)_{diag} U = \sum_k U^\dagger |k\rangle\langle k|U\rho U^\dagger|k\rangle\langle k|U. \tag{7.9}$$

Taking $|l\rangle \equiv U^\dagger|k\rangle$, we can rewrite the above equation as:

$$U^\dagger (U\rho U^\dagger)_{diag} U = \sum_l |l\rangle\langle l|\rho|l\rangle\langle l| = \Pi_l(\rho). \tag{7.10}$$

Substituting Eq. (7.10) in Eq. (7.8), we deduce that:

$$C_{min}(\rho) = \mathcal{D}(\rho, \Pi_{min}(\rho)) = \mathcal{Q}(\rho), \tag{7.11}$$

where $\Pi_{min}(\rho)$ is a projective measurement that minimizes $C(\rho)$. □

Thus, we conclude that C_{min} represents a discord-type quantum correlation. A similar result was found for the relative entropy of discord $\mathcal{Q}_r(\rho)$, where the basis-free quantum coherence is calculated by the relative entropy of coherence, i.e., $C^{free}(\rho) = \min_{U} C_{\mathcal{D}}(U\rho U^\dagger) = \mathcal{Q}_r(\rho)$ (YAO et al., 2015).

In the following, we present in the theorem 4 our results about the relation between the $C_{max}(\rho)$ and the measurement induced non-locality (MIN) . MIN was firstly proposed by (LUO; FU, 2011), which can be considered as a quantum correlation from a geometric perspective based on the local projective measurements from which one of the reduced states is left invariant (GUO, 2013). Consider a bipartite quantum state ρ shared by two parties a and b with respective system Hilbert spaces $\mathcal{H}_a$ and $\mathcal{H}_b$. In order to capture the

nonlocal effect of measurements on the state, it is required that the measurements do not disturb the local state $\rho_a = \mathrm{tr}_b\rho$. Hence, one can define the MIN of ρ (with respect to part A) as the maximal distance that a state to the set $\mathscr{L}$ of locally invariant quantum states, this is (LUO; FU, 2011):

$$\mathcal{N}(\rho) = \max_{\varrho\in\mathscr{L}} \mathcal{D}(\rho, \varrho) \tag{7.12}$$

where the locally invariant of ϱ means that $\varrho = \sum_k \Pi_k^a \rho \Pi_k^a$ and the maximum is taken over all local projective measurement $\Pi^a = \{\Pi_k^a\}$ which do no disturb ρ_a locally, that is, $\sum_k \Pi_k^a \rho_a \Pi_k^a = \rho_a$ where $\Pi^a(\rho) = \sum_k (\Pi_k^a \otimes \mathbb{I}_B)\rho(\Pi_k^a \otimes \mathbb{I}_B)$.

Theorem 4. *$C_{max}(\rho)$ is a measure of* MIN *(Measurement Induced Non-locality) if the restriction $\Pi_{max}^{(i)}(\rho_i) = \rho_i$ is imposed.*

Proof. The proof is straightforward if we follow the lines of reasoning of the proof of theorem 3. For the restriction of the corresponding farthest incoherent state is $\delta = \rho_{diag}$, C_{max} can be written as:

$$C_{max}(\rho) \equiv \max_U \mathcal{D}(\rho, U^\dagger (U\rho U^\dagger)_{diag} U). \tag{7.13}$$

Using Eq. (7.9) and taking $|l\rangle \equiv U^\dagger |k\rangle$, we obtain:

$$U^\dagger (U\rho U^\dagger)_{diag} U = \sum_l |l\rangle\langle l|\rho|l\rangle\langle l| = \Pi_l(\rho). \tag{7.14}$$

Substituting Eq. (7.14) in Eq. (7.13), we conclude that:

$$C_{max}(\rho) = \mathcal{D}(\rho, \Pi_{max}(\rho)) = \mathcal{N}(\rho), \tag{7.15}$$

where $\Pi_{max}(\rho)$ is a projective measurement that maximizes $\mathcal{N}(\rho)$ under the restriction $\Pi_{max}^{(i)}(\rho_i) = \rho_i$ for the reduce operator of the i-th part ρ_i. □

7.3 Trace-norm coherence

Let us now to focus in one specific distance, the trace-norm (the Schatten-1 norm). Thus, the coherence based in trace norm is defined by (BAUMGRATZ; CRAMER; PLENIO, 2014):

$$C_{\mathrm{tr}}(\rho) = \min_{\delta\in\mathcal{I}} \| \rho - \delta \|_1, \tag{7.16}$$

where $\| A \|_1 = \mathrm{tr}\sqrt{A^\dagger A}$ denotes the trace norm of the matrix A. This measure has been proved to be a coherence monotone and fulfill the conditions (C1)-(C3), for the cases where ρ is one-qubit (SHAO et al., 2015) or a X state (YU et al., 2016). Given two one-qubit

states $\rho = \frac{\mathbb{I}+\vec{r}.\vec{\sigma}}{2}$ and $\delta = \frac{\mathbb{I}+\vec{s}.\vec{\sigma}}{2}$, the trace-norm between ρ and δ can be expressed as (NIELSEN; CHUANG, 2000):

$$\mathcal{D}_{\text{tr}}(\rho,\delta) = \|\rho-\delta\|_1 = |\vec{r}-\vec{s}|. \tag{7.17}$$

Therefore, coherence is given by (BAUMGRATZ; CRAMER; PLENIO, 2014; SHAO et al., 2015):

$$C_{\text{tr}}(\rho) = \min_{\delta\in\mathcal{I}} \mathcal{D}(\rho,\delta) = \min_{\delta\in\mathcal{I}} \sqrt{(r_x-s_x)^2+(r_y-s_y)^2+(r_z-s_z)^2}. \tag{7.18}$$

For the incoherent states, we may take $s_x = s_y = 0$, whence coherence can be simplified as:

$$C_{\text{tr}}(\rho) = \min_{\delta\in\mathcal{I}} \sqrt{(r_x)^2+(r_y)^2+(r_z-s_z)^2} = \|\rho-\rho_{\text{diag}}\|_{\text{tr}} = \sqrt{r_x^2+r_y^2}. \tag{7.19}$$

Remarkably, for the case of one-qubit the closest incoherent state is $\delta = \rho_{diag}$ (BROMLEY; CIANCIARUSO; ADESSO, 2015; CHEN; FEI, 2017) and $C_{\text{tr}}(\rho)$ has the same expression as $C_{l_1}(\rho)$ for the case of one-qubit, reading:

$$C_{l_1}(\rho) = C_{\text{tr}}(\rho) = \sum_{i,j,i\neq j} |\rho_{ij}| = 2|\rho_{12}|. \tag{7.20}$$

Notice that C_{l_1} is a widely used quantifier of coherence induced by the l_1 matrix norm, $\mathcal{D}_{l_1}(\rho,\delta) = \|\rho-\delta\|_{l_1} = \sum_{i,j} |\rho_{i,j}-\delta_{i,j}|$ (BAUMGRATZ; CRAMER; PLENIO, 2014). On the other hand, for the case of the X states (see section 5.1), it has been demonstrated by (RANA; PARASHAR; LEWENSTEIN, 2016) that if ρ is a X state the nearest diagonal matrix to ρ_X in trace-norm is given by $\text{diag}(\rho_X)$. Hence, $C_{\text{tr}}(\rho_X) = C_{l_1}(\rho_X)$. Therefore, $C_{\text{tr}}(\rho_X)$ satisfy the conditions for a properly coherence measure.

Finaly, since r is a basis independent parameter and

$$C_{\text{tr}}(\rho) = 2|\rho_{12}| = \sqrt{r_x^2+r_y^2} = \sqrt{r^2-r_z^2}, \tag{7.21}$$

we obtain:

$$C_{min}(\rho) = 0,\ (r_z = r), \tag{7.22}$$

$$C_{max}(\rho) = r,\ (r_z = 0), \tag{7.23}$$

where C_{max} corresponds to the purity (NIELSEN; CHUANG, 2000). This result is in agreement with the results obtained by (SHI et al., 2017; STRELTSOV et al., 2016). Also, for any distance-based coherence monotone the maximal coherence achievable via unitary operations coincide with the corresponding distance-based purity monotone (STRELTSOV et al., 2016). On the other hand, $C_{min} = 0$ is an immediate result for one-qubit because is always possible to choose a local basis such that ρ is diagonal.

7.4 Quantum coherence under a decohering process

Incoherent Kraus operators showed in section 7.1 are of particular importance for decoherence processes, as already defined in this section, an incoherent operation ICPTP is a CPTP map, which always maps any incoherent state to another incoherent one (see the definitions A) and B) in section 7.1). The channels showed in section 2.6, namely, PF, BF, BPF, depolarizing, PD and AD damping channels are examples of such operations in regular computational basis $\{|0\rangle, |1\rangle\}$ (BROMLEY; CIANCIARUSO; ADESSO, 2015; HU et al., 2017; STRELTSOV; ADESSO; PLENIO, 2017). In particular, it has been shown that any bona fide distance-based measure of quantum coherence exhibits freezing phenomena for an even number of qubits, initialized in a particular class of states with maximally mixed marginals, and undergoing local independent and identical nondissipative flip channels (BROMLEY; CIANCIARUSO; ADESSO, 2015). In this section, we focus on two quantum channels, which are PD and AD. Their Kraus operators are presented in the Table 3.

	Kraus operators
PD	$E_0^s = \sqrt{1-p_s/2}\, I, E_1^s = \sqrt{p_s/2}\, \sigma_3$
AD	$E_0^s = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p_s} \end{pmatrix}, E_1^s = \begin{pmatrix} 0 & \sqrt{p_s} \\ 0 & 0 \end{pmatrix}.$

Table 3 – Kraus operators for PD and AD, where p_s is the decoherence probabilities for the qubit s.

Quantum coherence for the PD channel

Let us start by illustrating the behavior of a PD channel for a quantum system described by one-qubit. In this situation for a Markovian PD channel the decoherence probability is $p = 1 - \exp[-t\gamma]$, where γ is the rate of decay of the qubit. We find the evolution of the parameters $r_i(t)$ of one-qubit given by $\rho = \frac{\mathbb{I}+\vec{r}.\vec{\sigma}}{2}$ under a PD channel given by Table 3, where the time-depend correlation function $r_i(t)$ in terms of the initial value $r_i(0)$ are given by:

$$\begin{aligned} r_x(t) &= r_x(0)e^{-\gamma t}, \\ r_y(t) &= r_y(0)e^{-\gamma t}, \\ r_z(t) &= r_z(0). \end{aligned} \tag{7.24}$$

We can straightly get the quantum coherence (in the computational basis) as a function of time by using that:

$$C_{\text{tr}}(t) = \sqrt{r_x^2(t) + r_y^2(t)} = e^{-\gamma t} C_{\text{tr}}(0), \tag{7.25}$$

we find that in this case the coherence decay monotonically in the time and this channel do not allow freezing or sudden changes for trace-norm coherence. Moreover, the maximum of coherence is given by:

$$C_{max}(t) = C_{max}(0) e^{-\gamma t} \quad (r_z(t) = 0). \tag{7.26}$$

Quantum coherence for the AD channel

Let us now by analyzing the behavior of an AD channel for a quantum system described by one-qubit. We chose $p = 1 - \exp[-t\gamma]$ for the Markovian regime, where γ is the rate of decay of the qubit. Therefore the density matrix for the Kraus operator for AD channel, given by Table 3 evolve as:

$$\rho = \begin{pmatrix} \rho_{11}(0) + \rho_{22}(0)(1 - e^{-\gamma t}) & \rho_{12}(0) e^{-\gamma t/2} \\ \rho_{21}(0) e^{-\gamma t/2} & \rho_{22}(0) e^{-\gamma t} \end{pmatrix}, \tag{7.27}$$

and the quantum coherence (in a computational basis) given by Eq. (7.20) is defined as follows:

$$C_{\text{tr}}(t) = 2|\rho_{12}| = C_{\text{tr}}(0) e^{-\gamma t/2}, \tag{7.28}$$

we find that a similar behavior with the above case, the coherence decay monotonically in the time, sudden changes or freezing for trace-norm coherence are impossible for this channel for one-qubit. Additionally, the maximum of coherence is given by:

$$C_{max}(t) = C_{max}(0) e^{-\gamma t}. \tag{7.29}$$

7.5 Measuring non-Markovianity through quantum coherence

In this section, we will discuss a method of quantifying non-Markovianity through quantum coherence. Considering that most quantum information quantities are monotonic under local CPTP maps, we find that existing measures of non-Markovianity are based on the non-monotonic behavior of these quantities (BREUER; LAINE; PIILO, 2009; BREUER et al., 2016; RIVAS; HUELGA; PLENIO, 2010; LUO; FU; SONG, 2012). As we showed in section 6.1 one way to characterize the non-Markovianity is through $N_F(\Phi)$ in Eq. (6.3). Following the same line of thought, based on the monotonically decreasing behavior of quantum coherence measures under ICPTP maps, given by the condition (2a),

we have that for Markovian dynamics, it follows that $\frac{dC(\rho(t))}{dt} \leq 0$, where C is a proper quantum coherence measure. Thus, any violation of this monotonicity $\frac{dC(\rho(t))}{dt} > 0$ at any time t will provide an indication of non-Markovianity. From this non-monotonicity of quantum coherence measures $N_C(\Phi)$ is a quantifier of non-Markovianity and is given by (CHANDA; BHATTACHARYA, 2016):

$$N_C(\Phi) = \max_{\rho(0)\in\mathscr{C}} \int_{\frac{dC(\rho(t))}{dt}\geq 0} \frac{d}{dt} C(\rho(t))dt, \tag{7.30}$$

where the maximization is taken over all initial states $\rho(0)$ that belong to the set of all coherent states $\mathscr{C}$. Hence, the measure N_C represents one measure of the degree of non-Markovianity for dynamical maps that preserve incoherence and lead to the interpretation of the reservoir memory effect like a backflow of the maximum amount of quantum "coherence" on the initial state, after the state has been subject to a noisy channel for a certain time.

In order to quantitatively analyze of the quantifier of non-Markovianity $N_C(\Phi)$, in the following we are going to show one example. For this purpose, we use the operator-sum approach Eq. (2.36) to describe the interactions of a one-qubit with non-Markovian environments.

7.6 Example: Non-Markovian AD noise

The AD noise under non-Markovian effects is described by the following incoherent Kraus operators[2] (BELLOMO; FRANCO; COMPAGNO, 2007):

$$K_0 = |0\rangle\langle 0| + \sqrt{p}|1\rangle\langle 1|, \; K_1 = \sqrt{1-p}|0\rangle\langle 1|, \tag{7.31}$$

where

$$p \equiv p(t) = \exp(-\Gamma t)\left\{\cos(\tfrac{dt}{2}) + \tfrac{\Gamma}{d}\sin(\tfrac{dt}{2})\right\}^2,$$

with $d = \sqrt{2\gamma\Gamma - \Gamma^2}$. Here γ is the system-reservoir coupling constant and Γ is the decay rate of the qubit. The parameter Γ, defining the spectral width of the coupling, is related to the reservoir correlation time through $\tau_r \approx \Gamma^{-1}$. On the other hand, γ depends on the qubit relaxation time by $\tau_s \approx \gamma^{-1}$. In the weak coupling regime, i.e. for $\gamma \leq \Gamma/2$, $p(t)$ is monotonically decreasing. In this regime, the time of relaxation τ_s is greater than the reservoir correlation time τ_r. The behavior of $p(t)$ is essentially a Markovian exponential

[2] This Kraus operators fullfil the condition $K_n\mathcal{I}K_n^\dagger \in \mathcal{I}$. The proof is straightforward if we consider the incoherent state $\delta = \sum_i \delta_i|i\rangle\langle i|$ we obtain $K_0\delta K_0^\dagger = \delta_0|0\rangle\langle 0| + \delta_1|1\rangle\langle 1| \in \mathcal{I}$ and $K_1\delta K_1^\dagger = \delta_1(1-p)|0\rangle\langle 0| \in \mathcal{I}$.

decay controlled by γ. Whereas in the strong coupling regime $\gamma > \Gamma/2$, the reservoir correlation time is greater than the relaxation time and non-Markovian effects become relevant. Interestingly, the results for AD noise under Markovian regimen showed in Table 3 can be recovered if we take $p(t) = 1-\eta$, with η being the decoherence rate of AD channel (NIELSEN; CHUANG, 2000).

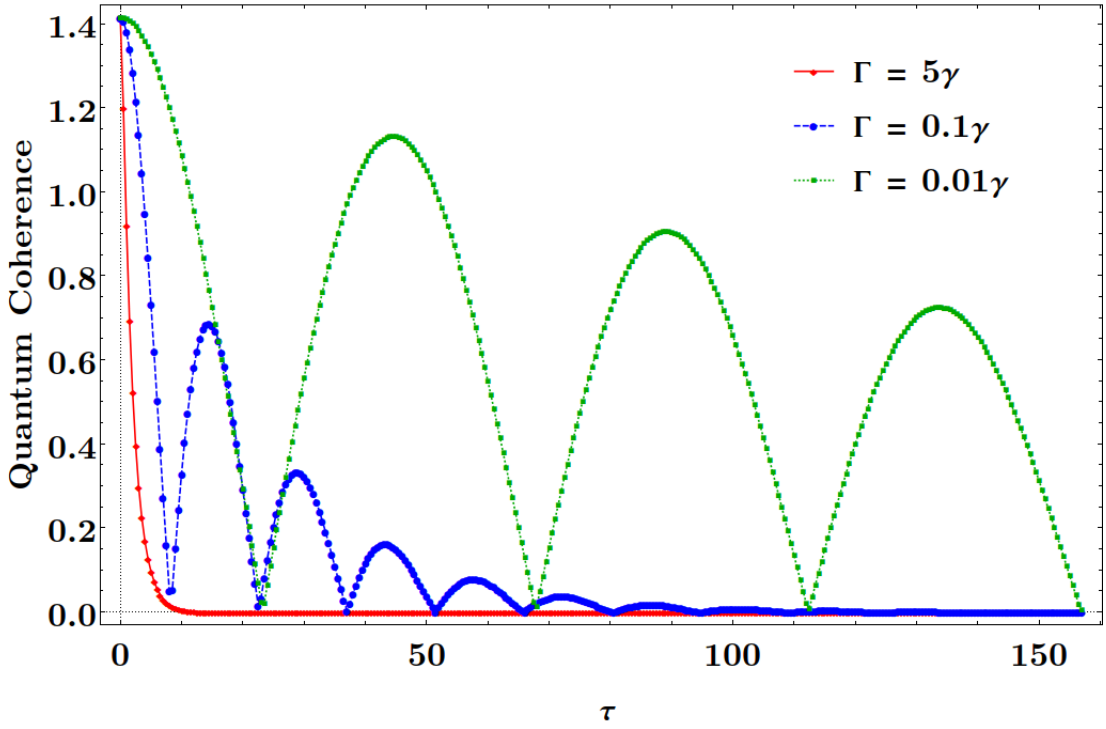

Figure 15 – Quantum Coherence as a function of $\tau = \gamma t$ for one-qubit system under the AD noise below non-Markovian effects. The initial state is one-qubit. The values for c_i are $c_1 = 0.65$, $c_2 = 0.55$, $c_3 = 0.2$.

To analyze the signature of non-Markovianity, we need to calculate the quantum coherence given by Eq. (7.20). In Fig. 15, we show the evolution of the coherence of one-qubit interacting with its non-Markovian surrounding given by the Eq. (7.31). The signature of non-Markovianity will be provided by the coherence, which has the advantages of avoiding the extremization procedures. The non-Markovian behavior can be witnessed in Fig. 15 for a AD noise, which shows the temporal evolution of the quantum coherence for several values of Γ, where we have strong regime of non-Markovian effect for $\Gamma = 0.01\gamma$ and weak regime of non-Markovian effect for $\Gamma = 0.1\gamma$, and $\Gamma = 5\gamma$ for Markovian regime, the initial state for one-qubit is described by $c_1 = -0.5$, $c_2 = 1$ and $c_3 = 1$. The oscillatory behavior present in the non-Markovian regimen can be seen as absorption and reemission of the quantum energy from the environment. Thus this information backflow is evident in the behavior of the quantum coherence showed in Fig. 15.

Let us now calculate the analytic solution for the degree of non-Markovianity N_C, for a AD noise. The function $p(t)$ presented in Eq. (7.31), can be rewritten in terms of the following parameters: $w = \frac{d}{2}$, $A\cos\phi = 1$ and $A\sin\phi = \frac{\Gamma}{d}$. With $\phi = \tan^{-1}(\frac{\Gamma}{d})$ ($\frac{-\pi}{2} \leq \phi \leq \frac{\pi}{2}$) and $A = \sqrt{(\Gamma/d)^2 + 1}$.Therefore, the function $p(t)$ is given by:

$$p(t) = e^{-\Gamma t} A^2 \cos^2(wt - \phi). \tag{7.32}$$

Using the definition of coherence for one-qubit Eq. (7.20), we calculate the coherence for a damping noise where the function $p(t)$ is in the form Eq. (7.32). Hence, we get:

$$C(t) = 2\sqrt{p}|\rho_{12}(0)| = 2|\rho_{12}(0)|Ae^{-\Gamma/2t}|\cos(wt - \phi)|, \tag{7.33}$$

where the period of oscillation of $|\cos(wt - \phi)|$ is given by $T = \frac{2\pi}{d}$. The behavior of the coherence for this case is schematically illustrated in the Fig. 16.

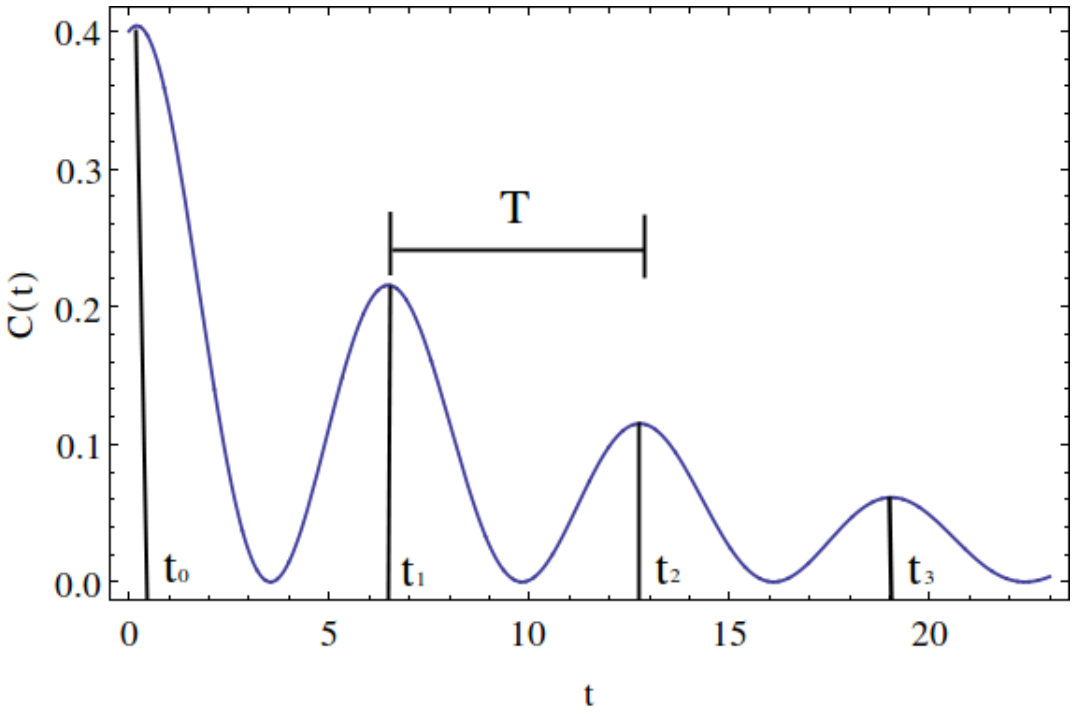

Figure 16 – Quantum Coherence as a function of t for an one-qubit arbitrary system under non Markovian noise.

To analyze the non-Markovianity, we need to calculate the measure N_C. In this way Eq. (7.30) can be written as:

$$N_C = \max_{\rho(0)} \sum_i \left[C(\tau_{i+1}) - C(\tau_i)\right], \tag{7.34}$$

where $\{(\tau_i, \tau_{i+1})\}$ represent the set of all intervals of time where $\frac{dC}{dt} > 0$. Thus for the quantum coherence given by Eq. (7.33) we have:

$$\{(\tau_i, \tau_{i+1})\} = \begin{cases} \{(0, t_0), (t_m - T/2, t_m)\} & 0 < t_0 \leq T/2, \\ \{(t_m - T/2, t_m)\} & -T/2 \leq t_0 \leq 0. \end{cases} \tag{7.35}$$

where $t_m = t_0 + mT$ $(m = 1, 2, 3..)$ and $t_0 = \frac{\phi}{\omega} = \frac{2}{d}\tan^{-1}(\frac{\Gamma}{d})$. In this order the measure of non-Markovianity based on quantum coherence is given by:

$$N_C = N_0 + \underset{\rho(0)}{\text{Max}} \sum_{m=1}^{\infty} C(t_m), \tag{7.36}$$

with

$$N_0 = \begin{cases} \underset{\rho(0)}{\text{Max}}[c(t_0) - c(0)] & 0 < t_0 \leq T/2, \\ 0 & -T/2 \leq t_0 \leq 0. \end{cases} \tag{7.37}$$

Taking the coherence for the AD noise Eq. (7.33) and using the previous equation, we obtain:

$$N_C = N_0 + 2A \sum_{m=1}^{\infty} e^{-\Gamma/2(t_0+mT)}, \tag{7.38}$$

with

$$N_0 = \begin{cases} 2(Ae^{-\Gamma t_0/2} - 1) & 0 < t_0 \leq T/2, \\ 0 & -T/2 \leq t_0 \leq 0, \end{cases} \tag{7.39}$$

here the maximization over the initial states is satisfied if we choose $\rho(0) = 1$. Consequently, the compact analytical expression for the quantifier N_C is given by

$$N_c = N_0 + \frac{2A}{e^{-\Gamma t_0/2}(e^{-\Gamma T/2} - 1)}, \tag{7.40}$$

where we used the relation $\sum_{m=1}^{\infty} e^{-\Gamma/2(t_0+mT)} = \frac{e^{-\Gamma t_0/2}}{e^{\Gamma T/2}-1}$. In the particular case when $t_0 \approx 0$ $(d \gg \Gamma)$, we get

$$N_C = \frac{2\sqrt{(\Gamma/d)^2 + 1}}{e^{\pi\Gamma/d} - 1} = \frac{2}{\pi}(\frac{d}{\Gamma}), \tag{7.41}$$

which correspond to the non-Markovian degree for strong non-Markovian regime.

8 CONCLUSIONS

In the first part of the thesis, a theoretical foundation for open quantum systems was developed. In the simplest case of the dynamics of an open system, the density matrix evolution is given by a quantum memoryless Markov master equation. Furthermore, the time-dependent Markovian master equation has been defined by the dynamical maps that fulfill the CP-divisibility condition. Then, we have presented a general framework to derive exact equations of motion for a non-Markovian open system, given by the TCL operator technique. In this thesis, two criteria have been considered for quantifying the degree of non-Markovian open system. The BLP measure that quantifies the distinguishability between quantum states and a different and non-equivalent criterion the RHP measure that identifies the non-Markovianity of quantum dynamics with the violation of a divisibility property of the family of dynamical maps. Both criteria are sufficient, but not necessary conditions to evaluate the non-Markovian features of a quantum process. We have seen for the analysis performed in this thesis that different quantities could be used for capturing a different aspect related to non-Markovian dynamics. A further study could investigate more general approaches for the characterization of non-Markovianity, for example, a recent proposed in (POLLOCK et al., 2018) where a universal framework to characterise arbitrary non-Markovian quantum processes is introduced. Note that this could represent a useful tool for a quantitative evaluation of non-Markovianity for dynamics undetected by non-Markovianity measures studied in this thesis. Finally, we have presented a brief revision of quantum correlations. Inspired by the fundamental relevance of quantum correlations into the quantum mechanics theory, we have described the different types of correlations that have been employed during the whole thesis.

In the second part of the thesis our results were presented. The results and applications that we have presented throughout the entire thesis have employed Wolfram Mathematica codes. The first results are the analytical expressions found for the trace-distance classical and total correlations for the case of two-qubit systems described by X states. In addition, we have shown the applicability of such correlations to investigate the dynamics of open quantum systems through the characterization of the pointer basis of an apparatus suffering either Markovian or non-Markovian decoherence. Since the non-Markovianity brings a flow of information from the environment back to the system during its evolution, the pointer basis has been found to emerge in a delayed time in comparison with the Markovian behavior. It is also remarkable to observe that, differently from the case of Bell-diagonal states, sudden transitions of entropic correlations for X states have been conjectured to display zero measure (PINTO; KARPAT; FANCHINI, 2013), which may compromise a precise characterization of the pointer basis. Our geometric

approach avoids this obstacle, since actual sudden changes are shown to be typical for general X states. This may have further implications in the characterization of quantum phase transitions through geometric classical correlations.

Secondly, we have introduced a unified framework based on generalized quantum, classical, and total correlation measures to characterize the non-Markovianity of local dynamical maps over multipartite quantum systems. This approach establishes sufficient conditions under which each class of correlation can be used to determine the degree of non-Markovian behavior. We illustrated our results for different master equation methods and for different sources of decoherence. We expect applications in experimental setups for which correlations may be accessible to the observer. In addition, the vanishing of entanglement for high-temperature regimes (WERLANG et al., 2010) or for distant neighbors within a composite system (MAZIERO et al., 2012), may also motivate the use of generalized correlations as a tool to characterize non-Markovianity. Further applications include the assessment of other approaches beyond Markovianity (see, e.g., Ref. (SHABANI; LIDAR, 2005)) and of additional axioms over correlation functions (see, e.g., Refs. (CIANCIARUSO et al., 2015; HU et al., 2012)). These topics are left for future research.

As a following topic, we have systematically studied the quantum coherence, employing the general distance-based approach as a coherence quantifier. In the context of the basis-free measure of coherence, we have presented the equivalence between of minimum amount of coherence and quantum discord for a general distance-based approach. Remarkably, we have defined a basis-free measure of coherence by the maximization of overall local unitary transformations and we have found that this quantity is exactly equivalent to the MIN. These correspondences enhance our understanding of the relation between different types of quantum correlations.

Furthermore, as a particular case of one-qubit, we have used the trace-norm coherence as a distance measure. We have found that the trace-norm coherence is equivalent to the l_1 matrix norm coherence, in agreement with the previous results found in (BROMLEY; CIANCIARUSO; ADESSO, 2015; CHEN; FEI, 2017). Since the study was limited to one-qubit for which the nearest diagonal state of ρ is given by its diagonal part, the identification of the nearest diagonal state is an open question for systems with two or more qubits (RANA; PARASHAR; LEWENSTEIN, 2016; BROMLEY; CIANCIARUSO; ADESSO, 2015; CHEN; FEI, 2017). In addition, we have analyzed the quantum coherence under PD and AD noises, and observed that neither channels allow freezing or sudden changes in trace-norm coherence. Nevertheless, more complex channels (CARUSO et al., 2014) or other possible candidates besides the trace distance, such as relative entropy (MODI et al., 2010; BAUMGRATZ; CRAMER; PLENIO, 2014), the Bures metric (SPEHNER; ORSZAG, 2013), or the Hellinger distance (CHANG; LUO, 2013) may allow these kinds of behaviors.

Finally, we have shown that under the allowed incoherent operation criteria, the monotonicity of the valid coherence measure may be affected by a partial backflow of the previously lost information of the system to the environment. This is similar to the cases of non-Markovian effects on the distinguishability between two different states and other quantum information measures (FANCHINI et al., 2014; PAULA; OBANDO; SARANDY, 2016; LORENZO; PLASTINA; PATERNOSTRO, 2013; BREUER et al., 2016; BREUER; LAINE; PIILO, 2009). Our numeric and analytical results for AD non-Markovian noise have suggested that coherence by trace-norm captures the non-Markovianity features of the system of one-qubit. Remarkably, from ours results an advantage of this method is that it is quantitatively less complicated, implying a simpler process of optimization and the relevant fact that one only needs one-qubit to characterize the non-Markovian dynamics. A future study investigating the non-Markovian behavior by basis-free measures of coherence would be interesting, where the process of optimization is avoided.

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Appendix

APPENDIX A – CLASSICAL MARKOV PROCESSES

The elements ω of the sample space Ω can be rather abstract objects. A random variable X is defined to be a map (BREUER; PETRUCCIONE, 2002)

$$X : \Omega \mapsto \mathbb{R}, \tag{A.1}$$

which assigns to each elementary event $\omega \in \Omega$ a real number $X(\omega)$. Given some ω the value

$$x = X(\omega), \tag{A.2}$$

is called a realization of X. A stochastic process is defined as a random variable whose statistical properties change in time. In mathematical terms, a stochastic process is a family of random variables $X(t)$ on a common probability space depending on a parameter $t \in T$. Corresponding to this definition, for each fixed t the quantity $X(t)$ is a map from the sample space ω into $\mathbb{R}$. A stochastic process can be regarded as a map

$$X : \Omega \times T \to \mathbb{R}, \tag{A.3}$$

which associates with each $\omega \in \Omega$ and with each $t \in T$ a real number $X(\omega, t)$. Keeping ω fixed, we call the mapping

$$t \mapsto X(\omega, t),\ t \in T, \tag{A.4}$$

a realization, trajectory, or sample path of the stochastic process.

A Markov process is a stochastic process $X(t)$ with a short memory, that is a process which rapidly forgets its past history (BREUER; PETRUCCIONE, 2002). A stochastic process $X(t)$, $t \geq 0$ taking values in a discrete set $\{x_i\}_{i\in\mathbb{N}}$ is characterized by a hierarchy of joint probability distributions of order n,

$$P_n = P_n(x_n, t_n; x_{n-1}, t_{n-1}; ....; x_1, t_1),$$

for all $n \in \mathbb{N}$ and the successive set of times $(t_n \geq t_{n-1} \geq ... \geq t_1 \geq 0)$, known as Kolmogorov hierarchic. The distribution P_n yields the probability that the process takes on the value x_1 at time t_1,the value x_2 at time t_2...and the value x_n at time t_n. A stochastic process $X(t)$ is said Markovian if satisfy (BREUER; PETRUCCIONE, 2002):

$$P(x_{n+1}, t_{n+1}|x_n, t_n; ...; x_1, t_1) = P(x_{n+1}, t_{n+1}|x_n, t_n), \tag{A.5}$$

where the conditional probability is defined by

$$P(x_{n+1}, t_{n+1}|x_n, t_n; ...; x_1, t_1) = \frac{P(x_{n+1}, t_{n+1}; ...; x_1, t_1)}{P(x_n, t_n; ...; x_1, t_1)}. \tag{A.6}$$

The equation (A.5) is the classical Markov condition which means that the probability for the stochastic process to take the value x_{n+1} at the time t_{n+1}, under the condition that it assumed values x_i at the previous time t_i, depend only on the last previous value x_n at time t_n. In this sense the process is a memoryless process (BREUER et al., 2016). A Markov process can be completely determined from the initial one-point distribution $P(x_0, t_0)$ and the conditional transition probability (BREUER; PETRUCCIONE, 2002)

$$T(x, t|y, s) \equiv P(x, t|y, s), \tag{A.7}$$

through

$$P_n(x_n, t_n; x_{n-1}, t_{n-1}; ....; x_1, t_1) = \Pi_{i=1}^{n-1} T(x_{i+1}, t_{i+1}|x_i, t_i) P(x_1, t_1), \tag{A.8}$$

and

$$P_1(x_1, t_1) = \sum_{x_0} T(x_1, t_1|x_0, t_0) P_1(x_0, t_0). \tag{A.9}$$

For a Markov process the transition probability has to obey the Chapman-Kolmogorov equation

$$T(x, t|y, s) = \sum_z T(x, t|z, \tau) T(z, \tau|y, s), \tag{A.10}$$

for $t \geq \tau \geq s$. Thus, a classical Markov process is uniquely characterized by a probability distribution for the initial states of the process and a conditional transition probability satisfying the Chapman-Kolmogorov equation. From of the Chapman-Kolmogorov equation, is possible obtain an equivalent differential equation (BREUER et al., 2016):

$$\frac{d}{dt} T(x, t|y, x) = \sum_z W_{xz}(t) T(z, t|y, s) - W_{zx}(t) T(x, t|y, s), \tag{A.11}$$

where $W_{zx} \geq 0$ represents the rate (probability per unit of time) for a transition to the state z given that the state is x at time t. An equation of the same structure in terms of the one-point probability distribution of the process:

$$\frac{d}{dt} P_1(x, t|y, x) = \sum_z W_{xz}(t) P_1(z, t) - W_{zx}(t) P_1(x, t). \tag{A.12}$$

APPENDIX B – THE NAKAJIMA-ZWANZIG EQUATION

B.1 Derivation of the Nakajima-Zwanzig equation

The aim of this Appendix is to derive the Nakajima-Zwanzig equation (3.5). The evolution equations for the relevant and irrelevant part of the density matrix are obtained applying the projection operators $\mathcal{P}$ and $\mathcal{Q}$ to Eq. (3.3) (BREUER; PETRUCCIONE, 2002). This yields [1]:

$$\frac{\partial}{\partial t}\mathcal{P}\rho(t) = \mathcal{P}\frac{\partial}{\partial t}\rho(t) = \alpha\mathcal{P}\mathcal{L}(t)\rho(t), \tag{B.1}$$

$$\frac{\partial}{\partial t}\mathcal{Q}\rho(t) = \mathcal{Q}\frac{\partial}{\partial t}\rho(t) = \alpha\mathcal{Q}\mathcal{L}(t)\rho(t). \tag{B.2}$$

On inserting the identity $\mathbb{I} = \mathcal{P} + \mathcal{Q}$ between $\mathcal{L}(t)$ and the density matrix ρ, we obtain:

$$\frac{\partial}{\partial t}\mathcal{P}\rho(t) = \alpha\mathcal{P}\mathcal{L}(t)\mathcal{P}\rho(t) + \alpha\mathcal{P}\mathcal{L}(t)\mathcal{Q}\rho(t), \tag{B.3}$$

$$\frac{\partial}{\partial t}\mathcal{Q}\rho(t) = \alpha\mathcal{Q}\mathcal{L}(t)\mathcal{P}\rho(t) + \alpha\mathcal{Q}\mathcal{L}(t)\mathcal{Q}\rho(t). \tag{B.4}$$

The solution of Eq. (B.4) for a given $\rho(t_0)$ at some initial time t_0 can be written as (BREUER; PETRUCCIONE, 2002):

$$\mathcal{Q}\rho(t) = \mathcal{G}(t,t_0)\mathcal{Q}\rho(t_0) + \alpha\int_{t_0}^{t} ds\mathcal{G}(t,s)\mathcal{Q}\mathcal{L}(s)\mathcal{P}\rho(s), \tag{B.5}$$

where $\mathcal{G}(t,s)$ is given by:

$$\mathcal{G}(t,s) = T_{\leftarrow}\exp\left(\alpha\int_0^t ds'\mathcal{Q}\mathcal{L}(s')\right). \tag{B.6}$$

Here the operator $T_{\leftarrow}$ denotes the chronological time ordering. The above equation satisfies

$$\frac{\partial}{\partial t}\mathcal{G}\rho(t) = \alpha\mathcal{Q}\mathcal{L}(t)\mathcal{G}(t,s), \tag{B.7}$$

with the initial condition $\mathcal{G}(s,s) = \mathbb{I}$. Now, substituting Eq. (B.5) into Eq. (B.3), we get the Nakajima-Zwanzig equation (BREUER; PETRUCCIONE, 2002):

[1] Is important to note that in this demonstration the density matrix ρ_E used in Eq. (3.4) is independent of time.

$$\begin{aligned}\frac{d}{dt}\mathcal{P}\rho(t) &= \alpha\mathcal{P}\mathcal{L}(t)\mathcal{G}(t,t_0)\mathcal{Q}\rho(t_0)+\alpha\mathcal{P}\mathcal{L}(t)\mathcal{P}\rho(t)\\ &+ \alpha^2\int_{t_0}^{t}ds\mathcal{P}\mathcal{L}(t)\mathcal{G}(t,s)\mathcal{Q}\mathcal{L}(s)\mathcal{P}\rho(s).\end{aligned} \tag{B.8}$$

This can be simplified if it is assumed that the odd moments of the interaction Hamiltonian with respect to ρ_E vanish:

$$\mathrm{tr}_E\{H_I(t_1)H_I(t_2)...H_I(t_{2n+1})\rho_E\}=0, \tag{B.9}$$

which leads to the relation

$$\mathcal{P}\mathcal{L}(t_1)\mathcal{L}(t_2)...\mathcal{L}(t_{2n+1})\mathcal{P}=0, \tag{B.10}$$

for $n = 0, 1, 2....$ Consequently the second term of Eq. (B.8) vanish, it follows that (BREUER; PETRUCCIONE, 2002):

$$\frac{d}{dt}\mathcal{P}\rho(t)=\alpha\mathcal{P}\mathcal{L}(t)\mathcal{G}(t,t_0)\mathcal{Q}\rho(t_0)+\alpha^2\int_{t_0}^{t}ds\mathcal{P}\mathcal{L}(t)\mathcal{G}(t,s)\mathcal{Q}\mathcal{L}(s)\mathcal{P}\rho(s). \tag{B.11}$$

B.2 Markov limit for the Nakajima-Zwanzig equation

The Eq. (B.8) can be expanded around t in powers of the memory time, this is in the width of the kernel $\tilde{K}(t,s)$. Clearly, for $\tilde{K}(t,s)\approx\delta(t,s)$ in the absence of memory effects the Markovian description is obtained (BREUER; PETRUCCIONE, 2002).

For a factorizing initial condition $\rho(t_0)=\rho_S(t_0)\otimes\rho_E$, this is $\mathcal{P}\rho(t_0)=\rho(t_0)$, such that $\mathcal{Q}\rho(t_0)=0$, the first term of Eq. (B.8) vanishes and assuming that, in general, any string containing an odd number of $\mathcal{L}$ between factors of $\mathcal{P}$ vanishes, Eq.(B.10), the term $\mathcal{P}\mathcal{L}_{tot}(t)\mathcal{P}=0$. Therefore, the Nakajima-Zwanzing equation (B.8) reduces to (BREUER; PETRUCCIONE, 2002):

$$\frac{\partial}{\partial t}\mathcal{P}\rho(t)=\int_{t_0}^{t}ds\tilde{K}(t,s)\mathcal{P}\rho(s). \tag{B.12}$$

The memory kernel $\tilde{K}(t,s)$ can be expanded in terms of the weak-coupling parameter between system and environment. For example, up to second order in the coupling strength α, we obtain:

$$\tilde{K}(t,s)=\alpha^2\mathcal{P}\mathcal{L}(t)\mathcal{Q}\mathcal{L}(s)\mathcal{P}+\mathcal{O}(\alpha^3). \tag{B.13}$$

Consequently, the equation of motion of second order for $\mathcal{P}\rho(t)$ is given by:

$$\frac{\partial}{\partial t}\mathcal{P}\rho(t)=\alpha^2\int_{t_0}^{t}ds\mathcal{P}\mathcal{L}(s)\mathcal{P}\rho(s), \tag{B.14}$$

where it is used the identity $\mathbb{I} = \mathcal{P} + \mathcal{Q}$ and the fact that $\mathcal{P}\mathcal{L}(t)\mathcal{P} = 0$. Introducing the explicit expressions for $\mathcal{P}$ and $\mathcal{L}(t)$, it follows that:

$$\frac{\partial}{\partial t}\rho_S(t) = \alpha^2 \int_{t_0}^{t} ds \mathrm{tr}_E[H_I(t), [H_I(s), \rho_S(s) \otimes \rho_E]]. \tag{B.15}$$

This equation is known as the Born approximation of the master equation (BREUER; PETRUCCIONE, 2002). Applying the Markov approximation in which the quantum master equation is made local in the time by replacing the density matrix $\rho_S(s)$ at the retarded time s with that at the present time $\rho_S(t)$. Moreover, the integration limit is pushed to infinity to get the Born-Markov approximation of the master equation. Physically, this implies that the bath correlation time τ_B is small compared to the relaxation time of the system, i.e $\tau_B \ll \tau_R$. Finally, in the rotating wave approximation rapidly oscillating terms proportional to $\exp[i(w' - w)t]$ for $w' \neq w$ are neglected, ensuring that the quantum master equation is in the Lindblad form (BREUER; PETRUCCIONE, 2002).

APPENDIX C – 1-NORM GEOMETRIC QUANTUM DISCORD FOR BELL-DIAGONAL STATES

In this Appendix we review the demonstration derived in (PAULA; OLIVEIRA; SARANDY, 2013) for an explicit formula of D_G for two-qubit Bell-diagonal states. The 1-norm geometric quantum discord is defined as:

$$D_1(\rho) = \min_{\rho_c \in \mathcal{C}} \| \rho - \rho_c \|_1, \tag{C.1}$$

where $\| X \|_1 = \mathrm{tr}[\sqrt{X^\dagger X}]$ is the 1-norm, ρ_c is an arbitrary classical-quantum state given by Eq. (4.15) and ρ represents the case of two-qubit Bell diagonal states, whose density operator is given by:

$$\rho^{Bell} = \frac{1}{4}(\mathbb{I}_4 + \sum_{i=1}^{3} c_i \sigma_i \otimes \sigma_i), \tag{C.2}$$

where $\mathbb{I}$ is the identity matrix, $\vec{c} = (c_1, c_2, c_3)$ is a three-dimensional vector and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is a vector formed by Pauli matrices. Assuming that the minimization in Eq. (C.1) is archived by a Bell-diagonal classical state ρ_c^{Bell}, which is described by:

$$\rho_c^{Bell} = \frac{1}{4}[\mathbb{I} \otimes \mathbb{I} + \vec{l}.(\sigma \otimes \sigma)], \tag{C.3}$$

where $\vec{l}$ represents a vector over the perpendicular classical axes in the tetrahedron of Bell- diagonal states (see Fig. 5). Therefore, $\vec{l}$ has the form $\vec{l}_1$, $\vec{l}_2$ or $\vec{l}_3$, with $l_i \in \mathcal{R}$ and $-1 \leq l_i \leq 1$. Replacing Eq. (C.2) and Eq. (C.3) in Eq. (C.1) is obtained:

$$D_1 = \min[\min_{l_1} f_1(l_1), \min_{l_2} f_2(l_2), \min_{l_3} f_3(l_3)], \tag{C.4}$$

where

$$f_i(l_i) = \| \frac{1}{4}(\vec{c} - \vec{l}_i).(\vec{\sigma} \otimes \vec{\sigma}) \|_1 = \sum_{p=0}^{1} \sum_{q=0}^{1} |\tau_{pq,i}|, \tag{C.5}$$

where the eigenvalues of the operator $(\vec{c} - \vec{l}_i).(\vec{\sigma} \otimes \vec{\sigma})/4$ are represented by $\tau_{pq,i} = [(-1)^p(c_i - l_i) - (-1)^{p+q}c_j + (-1)^q c_k]/4 \, (i \neq j \neq k)$. If we define $d_i = l_i - c_i$ and $d_\pm = c_k \pm c_j$, is possible obtain:

$$f_i(d_i) = (|d_i + d_+| + |d_i - d_+| + |d_i + d_-| + |d_i - d_-)/4, \tag{C.6}$$

which reaches the minimum value when $d_i = 0$, then $\min_{l_i} f_i(l_i) = \min_{d_i} f_i(l_i) = \max[|c_j|, |c_k|]$. Hence, using this result in Eq. (C.4), the 1-norm geometric discord is given by:

$$D_1 = c_0 = \text{int}[|c_1|, |c_2|, |c_3|], \tag{C.7}$$

with int$\{\cdot\}$ denoting the intermediate among the elements of the set $\{|c_1|, |c_2|, |c_3|\}$.

APPENDIX D – GEOMETRIC CLASSICAL AND TOTAL CORRELATIONS FOR BELL-DIAGONAL STATES

In this Appendix, we recall the main steps to derive the analytical expressions for the geometric classical and total correlations through Schatten 1-norm, Eq. (4.33) and Eq. (4.34) respectively, as demonstrated in (PAULA et al., 2014). As we have already mentioned in subsection 4.3.2.1, the general form of the two-qubit Bell-diagonal is the following:

$$\rho^{Bell} = \frac{1}{4}[\mathbb{I} \otimes \mathbb{I} + \vec{c}.(\vec{\sigma} \otimes \vec{\sigma})], \tag{D.1}$$

where $\mathbb{I}$ is the identity matrix on the subsystem, $\vec{c} = (c_1, c_2, c_3)$ is a three-dimensional vector and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is a vector formed by Pauli matrices. The eigenstates of ρ^{Bell} are the four Bell states Eq. (4.25) with eigenvalues given by Eq. (4.26). With the aim to obtain the optimal classical-quantum state $M(\rho)$, which will be required for the definition of G_T and T_T. The initial step is considered an alternative derivation of D_T to the approach presented in the previous Appendix. Thus, the 1-norm geometric quantum discord is given by:

$$D_T = \mathrm{tr}|\rho - M(\rho)|. \tag{D.2}$$

The operator $\rho - M(\rho)$ have four possible eigenvalues which are given by $\gamma_1 = \gamma_+$, $\gamma_2 = -\gamma_+$, $\gamma_3 = \gamma_-$, and $\gamma_4 = -\gamma_-$, where:

$$\gamma_\pm = \frac{1}{4}\sqrt{c^2 - \vec{\alpha}.\vec{u} \pm 2\sqrt{\vec{\beta}.\vec{u}}}, \tag{D.3}$$

with $c^2 = c_1^2 + c_2^2 + c_3^2$, $\vec{\alpha} = (c_1^2, c_2^2, c_3^2)$, $\vec{\beta} = (c_2^2 c_3^2, c_1^2 c_3^2, c_1^2 c_2^2)$, and $\vec{u} = (n_1^2, n_2^2, n_3^2)$. Thus, Eq. (D.2) can be written as:

$$D_T = \sum_{i=1}^{4} |\gamma_i| = 2[\gamma_-(\vec{u}) + \gamma_+(\vec{u})], \tag{D.4}$$

where $\vec{u}$ minimizes the function $f(\vec{u}) = \gamma_-(\vec{u}) + \gamma_+(\vec{u})$ under the conditions $u_1 + u_2 + u_3 = 1$ and $0 \leq u_i \leq 1$. Applying the Lagrange multipliers method, is possible to find that

$\vec{u} = \hat{\epsilon}_j (j = 1, 2, \text{or } 3)$, where $\hat{\epsilon}_j$ represents the unitary vector in a fixed c_j direction. Using this result in equation above, D_T is now given by:

$$D_G = \max\{|c_{j+1}|, |c_{j+2}|\}, \tag{D.5}$$

where the correlations c_{j+1} and c_{j+2} are defined by $c_{j+k} = c_{j+k(mod3)}$. Specifically, $c_{2+2} = c_{3+1} = c_1$ and $c_{3+2} = c_2$. Defining the maximum

$$c_+ = \max\{|c_1|, |c_2|, |c_3|\}, \tag{D.6}$$

the intermediate

$$c_0 = \text{int}\{|c_1|, |c_2|, |c_3|\}, \tag{D.7}$$

and the minimum

$$c_- = \min\{|c_1|, |c_2|, |c_3|\}, \tag{D.8}$$

values within the set $\{|c_1|, |c_2|, |c_3|\}$. Eq. (D.5) is minimized when j is such that $|c_j| = c_+$. Hence, the 1-norm geometric quantum discord is given by:

$$D_T = \max\{c_-, c_0\} = c_0 \tag{D.9}$$

Considering the unitary vector $\vec{n} = \pm\hat{\epsilon}_j$ that minimizes D_T in Eq. (4.32). The classical-quantum states Eq. (4.31) are described by:

$$M(\rho) = \frac{1}{4}(\mathbb{I} \otimes \mathbb{I} + c_j \sigma_j \otimes \sigma_j). \tag{D.10}$$

Moreover, the reduced density operators of the Bell diagonal states Eq. (C.2) correspond to the maximally mixed states, i.e., $\rho_A = \text{tr}_B \rho = \mathbb{I}/2$ and $\rho_B = \text{tr}_A \rho = \mathbb{I}/2$. Therefore, the product of the local marginals is given by:

$$\pi_\rho = \rho_A \otimes \rho_B = \frac{1}{4}(\mathbb{I} \otimes \mathbb{I}). \tag{D.11}$$

Consequently, $M(\pi_\rho) = \pi_\rho$. Then, the classical geometric correlation is given by:

$$C_G = \text{tr}|M(\rho) - M(\pi_\rho)|, \tag{D.12}$$

and the total geometric correlation is:

$$T_G = \text{tr}|\rho - \pi_\rho| = \sum_{i,j} |\lambda_{ij} - 1/4|, \tag{D.13}$$

where $\lambda_{ij} - 1/4$ are the eigenvalues of operator $\rho - \pi_\rho$. Finally, in terms of c_+, c_0 and c_-, these expressions can be written as:

$$C_G = c_+, \tag{D.14}$$

and

$$T_G = \frac{1}{2}[c_+ + \max\{c_+, c_0 + c_-\}]. \tag{D.15}$$