A renormalizable topological quantum field theory for gravity

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Abstract

In 1989 E. Witten, using traditional QFT techniques, develop an exact path integral representation to many classes of topological invariants. These QFTs, so-called Topological QFTs (TQFTs), share the property that all of their observables are metric-independent. In other words, the observables are global invariants classifying the topological and smooth structure of spacetime. In this sense, one could say, that TQFTs are examples of background independent and exactly soluble perturbative QFTs.

One of the most proeminent example perhaps is the four dimensional Topological Yang-Mills theory (TYM). This theory can be obtained by the BRST quantization of the Pontryagin invariant $\int \mathrm{Tr}\,(FF)$, instead of the tradition Yang-Mills action $\int \mathrm{Tr}\,(F \star F)$. The observables are known to be the Donaldson's polynomials, which classify the smooth structure of the underlying manifold. In particular, TYM theory is so symmetric that it has remarkably simple quantum properties. For instance, in the Landau gauge, it renormalizes, to all orders in perturbation theory, with only one (nonphysical) parameter and the theory is actually exactly soluble at tree-level (all quantum corrections vanish).

These remarkable properties led Witten to hypothesize if such a theory could describe an unbroken phase of General Relativity. In this thesis, we will propose a renormalizable TYM theory that can generate gravity via an explicity breaking of its topological BRST symmetry - thus fulfilling Witten's vision. In particular, we will consider the family of Lovelock-Cartan theories of gravity due to their generality and closer relation to the gauge structure.

Contents

Chapter 1

Introduction

The quantum mechanical behavior of the gravitational field remains mostly unknown even after ninety years of continuous scientific investigation. The main source of difficulty is the total lack of experimental data: quantum gravity (QG) corrections become dominant only at $\sim 10^{16}$ TeV, many orders of magnitude above the current output of particle accelerators[1].

Though it is very hard to justify the research in QG on practical grounds, a complete self-consistent theory should be able to solve or, at least, enlighten us about major open problems in theoretical physics. Some of these includes spacetime singularities, information loss, the nature of dark energy, chronology protection, cosmic and topology censorship, locality, extra dimensions, etc.

In contrast, the other three forces of Nature manifest their full quantum mechanical behavior in much lower energy scales, well within the reach of contemporany accelerators. For these, an extremely precise model for their description could be formulated.

The Standard model of Particle Physics, or Standard Model (SM), for short, describes the strong and electroweak force as quantum fields over spacetime with gauge symmetry $SU(3)_c \times SU(2)_L \times U(1)_Y$[2]. In particular, at $\sim 10^2$ GeV the electroweak sector $SU(2)_L \times U(1)_Y$ suffers a spontaneous symmetry breaking, via the Higgs mechanism, to its Abelian subgroup $U(1)_{em}$. The result is a splitting in the electromagnetic force, mediated by the massless photon γ, and the weak force, mediated by the massive bosons $W^{\pm}$ and Z^0 at energies lower than

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[1] For comparison, the current output of the Large Hardron Collider is $\sim 10 \times 10^1$ TeV.

[2] $SU(3)_c$ is the color symmmetry of strong interactions. $SU(2)_L$ is the weak isospin symmetry of left-handed fermions and $U(1)_Y$ is the weak hypercharge.

this threshold[3]. At about $\sim 10^2$ MeV, hardronization takes place and the $SU(3)_c$ sector gets confined inside hadrons.

In this Quantum Field Theory (QFT) framework, we can predict the strength of electromagnetic interactions, for instance, with a precision of ten parts per billion (10^{-8}), making Quantum Electrodynamics the most precise theory of all Sciences.

Besides the lack of experimental data, the apparent clash between the QFT framework and gravity also sources great theoretical difficulties in the QG research. On one hand, QFT describes the behavior of quantum field over a background manifold[4]. In other words, it is a background dependent framework that needs, for instance, a fixed background metric. On the other hand, we will see in chapter 4 that the most important lesson we learned from General Relativity (GR) was the background independence principle. In other words, that the role of gravity is to predict the features of spacetime, not to assume them *à priori*. For that reason, the metric must not be fixed in a gravitational theory, but dynamical, given by an action principle.

If this clash is just apparent or deeply rooted is still an open debate. The most tradicional program, however, is to give up background independence in favor of a perturbative QFT description. The reason is much more historical than evidence based. In fact, it is not clear at all if this program results in a sensible physical description. This is the so-called problem of $d = 4$ perturbative QG. Maybe a brief review is due.

In the path integral approach to quantum GR, for instance, one tries to make sense of a sum over all possible inequivalent geometries or, more precisely, a sum over the functional space of metrics modulus diffeomorphisms[5]. The usual program is to split the metric field into a simple sum of a fixed background $\bar{g}_{\mu\nu}$ and a massless symmetric field $h_{\mu\nu}$. In this way, gravity can be treated as a perturbative QFT for $h_{\mu\nu}$. Again, the presence of $\bar{g}_{\mu\nu}$ jeopardizes the background independence principle. But not only that, in mid seventies it was shown

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[3] At $\sim 10^2$ GeV the Higgs field attains a nontrivial vacuum expectation value (vev). This creates all sorts of new interactions and possibly mixing among the fields that were coupled to it. In particular, this is what happens among the $SU(2)_L$ and $U(1)_Y$ gauge fields, respectively, W_1, W_2, W_3 and B. Due to the nontrivial vev of the Higgs field, bosons W_1 and W_2 coalesce into two massive bosons, namely, $W^{\pm} = (W_1 \pm iW_2)/\sqrt{2}$, while bosons W_3 and B coalesce into the massive $Z^0 = -\sin(\theta)B + \cos(\theta)W_3$ and the massless $\gamma = \cos(\theta) + \sin(\theta)W_3$, where θ is the so-called mixing angle. The final result is the electromagnetic field as an $U(1)_{em}$ gauge field coupled nonminimally to the massive weak field.

[4] A manifold is, in general, a globally complicated space which locally looks very simple. Indeed, it can be seen as a generalization of the Euclidean space in which the tools of Differential Calculus apply only locally. Mathematically, it is defined as a topological space endowed with a differentiable atlas. We will exclusively consider manifolds with C^{∞}-atlases, also called smooth structures.

[5] A diffeomorphism is an isomorphism between manifolds, i.e., a differentiable and invertible map that preserves topological and smooth structure features. Manifolds connected by a diffeomorphism share the same global invariants and are physically indistinguishable from each other.

that this program is nonpredictive: it is perturbatively nonrenormalizable for GR dynamics[6] or nonunitary for higher derivatives (HD) Lagrangians[7].

Many other approaches were formulated in these ninety years of QG research. They, however, can be divided in basically two "world views". One understands QG as a quantum theory of spacetime, i.e., that gravity is fundamentally a spacetime property and therefore it must be quantized in an explictly background independent manner. Examples are the spinfoam formulation of Loop Quantum Gravity [10], Causal Dynamical Triangulations [11], Causal Sets [12], Tensor Models [13], etc.

Other approaches quantize gravity in such a way that background independence is lost - at least explictly - and there is no clear step on how to recover it. Even more drastically, some assume that spacetime properties are emergent, i.e., that gravity is fundamentally something else. In these approaches, geometrodynamics should be recovered in some limit, of course. One of most proeminent example of such model is the strings hypothesis [14, 15]. It, however, aims a far greater goal of a Theory of Everything. More conservative approaches includes (the already mentioned) HDQG [16], Asymptotic Safe Quantum Gravity [17], Supergravities [18], Hořava-Lifshitz [19], induced gravities [20], etc.

The scenario that will be analysed in the chapter 5 of this thesis unites the best of both worlds. Namely, that the quantum mechanical behavior of the gravitational field could be described by a perturbative renormalizable QFT that is also background independent - at least in some particular sense.

The idea is that a TQFT is such a theory, describing gravity at a trans-Planckian[8] regime. Here, the reader should be careful in realizing the paradigm shift. At these energies, gravity would not be about geometry anymore, since TQFTs describes no local degrees of freedom. This, however, should not be a surprise. Virtually every QG approach give up on geometrodynamics. For the same reason, there are no gravitons either. Again, no surprises. Particles are background dependent concepts.

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[6] In 1974 G. 't Hooft and M. Veltman showed that at 1-loop GR has divergences proportinal to R^2 and $R_{\mu\nu}R^{\mu\nu}$ [1]. Therefore the pure theory is 1-loop renormalizable (since it is Ricci flat) but nonrenormalizable in the presence of minimally coupled scalar fields. This result was later extended to spinor, vector and tensor fields [2–5]. The hope was that 2-loop corrections could cancel these divergences. In 1985, M. H. Goroff and A. Sagnotti showed that this was not the case [6]. In fact, the 2-loop divergences were worse, proportional to $R_{\alpha\beta}{}^{\mu\nu}R_{\mu\nu}{}^{\sigma\lambda}R_{\sigma\lambda}{}^{\alpha\beta}$, rendering GR nonrenormalizable even without matter field couplings.

[7] In 1977, K. S. Stelle introduced a class of gravity theories that were renormalizable to all orders in perturbationn theory. They, however, were plagued with ghost fields appearing in their physical spectrum. To be fair, until today there is still an open debate about whether or not these ghosts can be tamed [7–9]. Thus, HDQG still is an active field of research.

[8] This word means beyond Planck energy, i.e., energies higher than 10^{16} TeV.

TQFTs, however, describe global degrees of freedom. In this sense, in our scenario, QG would be reduced to the knowledge of global invariants of spacetime, nothing else. In particular, invariants that classify the smooth structure, known as the Donaldson polynomials.

The local degrees of freedom of gravity and thus geometrodynamics could only be generated by a breaking of the topological symmetry. It is reasonable to expect this to happen at Planck's energy. The breaking mechanism, however, will not be tackle in this thesis. Our work will be restrict to the analysis of the physical consistency of this scenario. In particular, if such TQFT is stable under quantum corrections and thus renormalizable to all orders. The algebraic renormalizability technique will be employed to this aim.

Chapter 2 is dedicated to a review of the concepts and tools that will be used throughout this thesis. This is done in the context of Yang-Mills theories for mainly two reasons. The first is that Yang-Mills theories are fairly well understood by most theoretical physicists. So, hopefully, the reader can start this thesis feeling confortable. The second, and more practical, was that the BRST construction adopted, hopefully, will make chapter 3 more natural: from the perspective of the BRST structure, it is more easy to comprehend TYM theories as natural generalizations of Yang-Mills theories.

Chapter 3 is dedicated to TQFTs with strong emphasis on TYM theories. In particular, their remarkable quantum properties in the self and anti-self-dual Landau [(A)SDL] gauges. Here, two novel results will be exposed i) the fact that, in the (A)SDL gauges, TYM theories renormalize with only one (unphysical) parameter, not four as stated in [21, 22] and; ii) that they are actually tree-level exact.

Chapter 4 is dedicated to gravity. We will start with the GR, putting particular emphasis on the background independence principle. We will argue that this principle leads to more general theories of gravity such as the Einstein-Cartan theory. Further, we will introduce the gauge theoretical approach to gravity, as it represents a more suitable framework for our purposes. This will culminate on the Lovelock-Cartan-(Sciama-Kibble) family as the most general theories of gravity that envolve nonvanishing torsion.

In chapter 5, we will propose the TYM theory that can generate the Lovelock-Cartan family via the explicit breaking of its topological BRST symmetry. In particular, we will present its renormalizability properties such as the most general counterterm, quantum stability, z-factors, etc. This is the main novel content of this thesis.

Finally, in chapter 6 the reader will find the conclusions and perspectives of this work.

Chapter 2

Yang-Mills theories

2.1 Introduction

In 1954, physicists C. N. Yang and R. Mills generalized the ideas of the electromagnetic $U(1)_{em}$ gauge symmetry to a more general, non-Abelian gauge group [23]. The objective was to develop a consistent QFT that could correctly describe the quantum dynamics of hadrons - particles that compose the atomic nuclei. The Yang-Mills (YM) theories, however, immediately faced severe difficulties.

The first one perhaps was the apparent masslessness of the gauge field. This problem was immediately pointed out by W. Pauli and rests on the fact that the gauge symmetry forbids the presence of a mass term for the gauge field in the action functional. In other words, the perturbative framework dictates that the fundamental excitations of the YM field must be massless vector bosons. Matter coupled to this field would thus experience a long range force, well beyond the typical nuclei distance. This was, of course, in direct contradiction to experimental data.

The second problem was the general feeling that the QFT framework was inherently ill-defined due to the inevitable presence of infinities as soon as one left the lowest order in perturbation theory. The physical meaning of the renormalization precedure was not yet known. In despite of the sucessful application in QED, it was believed that this procedure was a trick. A trick to hide the inconsistencies of QFT under a rug.

It was also believed that the perturbative QFT framework could not possibly describe fundamental interactions because these become infinitely strong at higher energies: a property known as a Landau pole. This general disbelieve was specially true among the fathers of QFT itself, namely R. P. Feynman, J. Schwinger, F. Dyson and others, making it even harder to be overcome.

Only in 1960 the first step towards the general acceptance of the YM theories was taken. Physicist Y. Nambu discovered, in the context of the BCS[1] theory for superconductivity, that new massless states could appear in the physical spectrum of gauge theories if one of its global symmetries was spontaneously broken[2] [24].

J. Goldstone subsequently elucidated Nambu's work in [25] and, together with A. Salam and S. Weinberg, convencily extended the results to a relativistc QFT framework [26]. These are now known as the Goldstone theorem and the resulting Nambu-Goldstone bosons [27].

In 1961 Schwinger was the first to realize that the Goldstone theorem did not exactly apply to local gauge symmetries [28]. In other words, that a spontaneous breaking of the local gauge invariance not necessarily led to propagating massless states. Nonetheless, it was P. Anderson, in 1962, the first to create a mechanism in which massive states could actually arise [29]. However, Anderson's work was in a nonrelativistic framework.

The relativistic case was develop in 1964 by three independent groups: i) P. Higgs [30]; ii) R. Brout and F. Englert [31] and; iii) G. Guralnik, C. R. Hagen and T. Kibble [32]. This mass generation mechanism is now known as the Higgs mechanism [27]. It precisely describes how a gauge theory coupled with a scalar field can aquire mass by "absorbing" the would-be Nambu-Goldstone bosons. This process exactly occurs when one of the gauged symmetries is partially or totally broken due to this scalar field, known as the Higgs field, attaining a nontrivial vev.

The Higgs mechanism solved the mass problem of YM theories. In particular, the Higgs field excitations were the latest particles of the SM to be experimentally verified. This was achieved, independently, by the CMS and ATLAS collaboration in 2012 [33, 34]. Shortly after, in 2013, Higgs as well as Englert were awarded the 2013 Nobel Prize in Physics.

Another important step occured in 1967, when Weinberg in [35] and Salam in [36] incorporated the Higgs mechanism to S. Glashow's unified model of weak and electromagnetic interactions [37]. In the Glashow-Salam-Weinberg (GSW) electroweak model, the $W^\pm$ and Z^0 massive bosons as well as the massless photon γ could be seen as the result of a partial spontaneous symmetry breaking, via the Higgs mechanism, of a $SU(2)_L \times U(1)_Y$ YM theory. In particular, the theory predicted a new kind of weak interaction among matter fields. An interaction in which the electric charge remained unchanged; very much like electromagnetism. The experimental discovery of the weak neutral current in 1973 confirmed this prediction [38]. For all of these contributions GSW were jointly awarded the 1979 Nobel Prize in Physics.

In despite of all of these results, the scientific community remained largely ignoring YM theories only until 1971. The turning point came in a series of two papers by 't Hooft in

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[1]BCS stands for Bardeen-Cooper-Schrieffer.

[2]A symmetry is said to be spontaneosly broken when it is present in the action functional but not in the vacuum state of the theory.

which he worked out their renormalization for the massless and massive case, including the GSW electroweak model [39, 40]. Particular importance should be given to the general method of regularizing gauge theories, named dimensional regularization, developed by him and Veltman, his Ph.D. advisor, in [41]. But which was also independently developed by J. J. Giambiagi and C. G. Bollini in that same year [42]. QFT was finally reborn as a consistent framework. For their work 't Hooft and Veltman were awarded the 1999 Nobel Prize in Physics.

The final and definite step was taken in 1973, when D. Gross, F. Wilczek and, independently, D. Politzer evaluated the β-functions of several YM theories [43, 44]. They discovered that they are always negative defined. In other words, that YM theories always exhibit asymptotically free behavior for large scales of field momenta, i.e., in their ultraviolet (UV) limit. This remarkable and very unexpected feature meant that YM theories did not possess Landau poles and thus the QFT framework could once again be seen as able to actually describe fundamental interactions.

The $SU(3)_c$ YM theory in particular remained asymptotically free even when coupled to up to 16 different fermions in the fundamental representation of this gauge group. This led Gross and Wilczek to propose it, with chiral flavor symmetry[3] $SU(3)_L \times SU(3)_R$, as the QFT for the strong interactions. In an effort to describe hadrons as quarks interating via gluons, similar conclusion was reached by H. Fritzsch, Gell-Mann and H. Leutwyler in [47] at that same year: QCD was then born. For the discovery of asymptotic freedom in the theory of strong interactions Gross, Wilczek and Politzer were jointly awarded the 2004 Nobel Prize in Physics.

Asymptotic freedom also meant that the infrared (IR) limit of YM theories was highly nonperturbative. In this complicated dynamics, we suppose that quarks and gluons condensate in a process known as color confinement [48–50]. This completely changes the QCD vacuum, generating all kinds of bound states: mesons, baryons, gluball, etc. For example, if the nonperturbative QCD vacuum breaks the chiral flavor symmetry $SU(3)_R \times SU(3)_L$ into its diagonal subgroup $SU(3)_V$, we can describe all eight pseudoscalar mesons as Nambu-Goldstone bosons.

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[3]Actually, the full chiral global symmetry is $U(3)_L \times U(3)_R$. It decomposes as $SU(3)_L \times SU(3)_R \times U(1)_V \times U(1)_A$. The subscript letter L, as we already explained, means that this group acts only on left-handed fermions. Letter R means this sector only acts on right-handed ones. Letter V stands for vectorial, this U(1) copy does not distinguish between left or right-handed fermions and it is related to the conservation of baryon number. On the other hand, letter A stands for axial and this U(1) copy does distinguish between left and right. The latter does not define any quantum number though, since it is anomalous; related to the U(1) problem of YM theories [45]. Anyhow, the chiral symmetry is only approximate since quarks top, down and strange do have nonvanishing masses. Nonetheless, these are very small when compared to Λ_{QCD}. This philosophy is part of M. Gell-Mann's eightfold way in describing hadrons via the representations theory of the SU(3) group [46], to which he was awarded the 1969 Nobel Prize in Physics.

Unfortunately, an analytic proof of color confinement is still lacking[4]. In fact, little is rigorously known about the IR regime of YM theories. We actually do not even have a precise definition of a quantum gauge theory in four dimensions. This challenge, together with the existence of a mass gap[5], constitute one of the Millenium Problems in Mathematics, as stated by the Clay Mathematics Institute, which offers a bounty of one million US dollars for its solution.

In this thesis, however, we will pretend that such hard facts do not exist and we will move on as if the perturbative framework worked flawlessly everytime.

2.2 Mathematical framework

The YM theories and gauge theories in general are not only physically but mathematically beatiful as well. This is because they have a natural geometrical interpretation in the language of fiber bundle theory. For instance, consider a principal G-bundle $\pi : P \rightarrow M$ where G is the structure group, π the projection, P the total space and M the base space[6]. Also, consider a G-connection 1-form ω on P, a local section σ over $x \in M$ as well as the Lie algebra $\mathfrak{g}$ of G. In this enlarged geometrical arena, the gauge field $A^{ab}_{\mu}(x)$ can be seen as the components of a $\mathfrak{g}$-valued 1-form A on $x \in M$, which is itself the result of ω being locally "pulledback" by σ from P to M. Mathematically,

$$A = \sigma^* \omega , \tag{2.1}$$

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[4]Confinement has been solved in other contexts, however. For example, it was discovered that the strongly coupled regime of four-dimensional $\mathcal{N} = 2$ Super-Yang-Mills theory with gauge group SU(2) is dual to a theory of weakly coupled monopoles. Confinement could then be exactly described by the physics of monopole condesation [51]. Surprisingly, this result also had deep impact on Mathematics, Differential Topology in particular, with the discovery of the Seiberg-Witten invariants. Confinement can also be described in $d = 3$ GSW electroweak model by considering nonperturbative contributions coming from instantons [52].

[5]A mass gap is an energy difference between the QCD vacuum and its first excited state. Its existence relies on this difference being strictly positive. In other words, it relies on the first excited state having a lower mass bound. This is, of course, in direct relation with color confinement: if gluballs exists as a consequence of color confinement, they are massive and such a mass gap is expected.

[6]If the reader is not familiar with this concept, the author recommends references [53, 54] as introductory, [55, 56] as intermediated and [57] as advanced. It can also be added that, loosely speaking, a fiber bundle can be seen as a larger geometrical arena that locally looks like a Cartesian product $P = M \times G$, but globally it usually has a much more complicated topological structure. In our particular context, the construction of a fiber bundle structure over spacetime can be seen, even more loosely speaking, as the process of adding extra, nonphysical, dimensions to M, provided by the structure group manifold G, that allow us to geometrize the force described by the gauge field. This idea is very reminiscent to T. Kaluza and O. Klein hypothesis of a fifth dimension that allow us to unify gravity and electromagnetism in a single geometrical setting [58]. Comments on the advantages of this formalism will be made before the end of this section.

where σ^* denotes the pullback by σ,

$$A = A^{ab}{}_\mu(x)J_{ab} \otimes dx^\mu \,, \tag{2.2}$$

dx^μ is a local basis on $x \in M$ and J_{ab} is a local basis on G near its unity, i.e., the generators of $\mathfrak{g}$ satisfying

$$[J_{ab}, J_{cd}] = f_{abcd}{}^{ef}J_{ef} \,, \tag{2.3}$$

where $[\ ,\ ]$ is a Lie bracket. More simply stated, A can be seen as a local representation of the globally defined connection form ω. Similar occurs with the curvature 2-form Ω of ω given by

$$\Omega = d_P\omega + \omega \wedge_P \omega \,, \tag{2.4}$$

where $\wedge_P$ is the wedge product and d_P the exterior derivative on P. It can also be "pulledback" to M, resulting in

$$\sigma^*\Omega = dA + A \wedge A \,, \tag{2.5}$$

where $\wedge$ is the wedge product and d the exterior derivative on M. We say that this $\mathfrak{g}$-valued 2-form on M is the curvature F of A,

$$F = dA + A \wedge A \,. \tag{2.6}$$

It can be expanded as

$$F = \frac{1}{2}F^{ab}{}_{\mu\nu}(x)J_{ab} \otimes (dx^\mu \wedge dx^\nu) \,, \tag{2.7}$$

and its components $F^{ab}{}_{\mu\nu}(x)$ are the usual YM field strength

$$F^{ab}{}_{\mu\nu}(x) = \partial_\mu A^{ab}{}_\nu(x) - \partial_\nu A^{ab}{}_\mu(x) + f_{cdef}{}^{ab}A^{cd}{}_\mu(x)A^{ef}{}_\nu(x) \,, \tag{2.8}$$

that we are so used to. Moreover, the curvature form Ω satisfies, as any curvature must, Bianchi identity

$$D^\omega\Omega = 0 \,, \tag{2.9}$$

where $D^\omega \equiv d_P + [\omega,\ ]$ is the exterior covariant derivative on P. This identity can also be "pulledback" to M by means of section σ, i.e., it can be locally represented by

$$DF = 0 \,, \tag{2.10}$$

where $D \equiv d + [A,\ ]$ is the exterior covariant derivative on M.

So, this will be the basic language that we will adopt in this thesis. The reason for this choice is fourfold: i) as already mentioned, this is the natural mathematical framework of gauge theories; ii) ω has a global meaning. Thus, understanding A as its local representation has greater interest when topology matters (as it does); iii) treating gauge theory geometrically represents an obvious step towards its connection to graviy and; iv) aesthetics: equations looks much more clean. In what follows, we will introduce some more concepts of gauge theory that are of much importance.

2.2.1 Gauge transformations

The group of gauge transformations $\mathcal{G}$ is defined as the group of equivariant automorphisms on P which induces the identity on M. An automorphism $f : P \to P$ is a diffeomorphism on P to itself. Note that this f induces an automorphism $f' : M \to M$ on M given by $f'(\pi(p)) = \pi(f(p))$ where $p \in P$. A gauge transformation is then an automorphism such that: i) $f(pg) = f(p)g$ for all $p \in P$ and $g \in G$ and; ii) $f' = \mathbb{1}_M$. Schematically,

$$
\begin{array}{ccc}
P & \xrightarrow{\ f\ } & P \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\
M & \xrightarrow[\ f'\ ]{} & M
\end{array}
$$

As a consequence, if ω is a connection on P then so it is its pullback $f^*\omega$ by f. This is the global, geometrical concept of a gauge transformation. Surely, it is more abstract than the reader might be used to. Nonetheless, one can still see it carries our familiar notion of what a gauge transformation is: a change of fields ($\omega \xrightarrow{f} f^*\omega$) that leaves the spacetime intact ($M \xrightarrow{\mathbb{1}} M$). Moreover, this definition is well suited to work with nontrivial bundles, i.e., when topology matters, due to its independence on local sections or trivializations.

If the reader remains unconfortable, the usual definition of a gauge transformation can be obtained if we think of f as acting on sections. For instance, consider a new section $\bar\sigma = f \circ \sigma$ in P to pullback ω to M resulting in

$$\bar{A} = \bar\sigma^*\omega \ . \tag{2.11}$$

Therefore, a change of sections $\sigma \xrightarrow{f} \bar\sigma$ due to the gauge transformation $f \in \mathcal{G}$ will amount to a change

$$\bar{A} = f^{-1}Af + f^{-1}df \ , \tag{2.12}$$

in the local representation A of ω. Equation (2.12) is, of course, the well-known gauge transformation of the YM gauge field that we are used to.

2.2.2   Observables

Moving on, the definition of a connection ω on P allows us to introduce the notion of parallel transport. As we know, there is no canonical way to compare vectors lying on different fibers of a vector bundle over M. But if we pick a connection and a path, there is a canonical way to drag a vector from one fiber to another. For instance, consider such a path the smooth loop $c(t) : [0, 1] \to$ M based at $x \in$ M. Also consider a point $p \in$ P in the fiber π^{-1} above x, i.e., $\pi(p) = x$. A connection ω on P then defines an unique horizontal lift of $c(t)$, given by $\tilde{c}(t) : [0, 1] \to$ P, such that $\tilde{c}(0) = p$. In other words, given a path on M, we can lift it up to the fibers above spacetime and the result is uniquely selected by the connection form ω. The lifted curve $\tilde{c}(t)$, however, is not, in general, closed. Its end point $\tilde{c}(1)$ will not, in general, be p but rather some other point $q = pg$; $g \in$ G in the fiber $\pi^{-1}(x)$. See figure 2.1 for a more visual description. This g is what we call the holonomy of ω around c, which is usually denoted
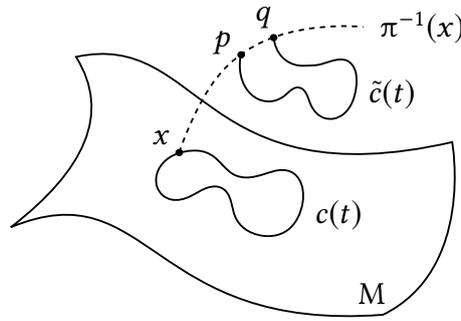
Figure 2.1 The curve $c(t)$ on M with endpoints $c(0) = c(1) = x$ and the the lifted curve $\tilde{c}(t)$ on P with endpoints $\tilde{c}(0) = p \neq \tilde{c}(1) = q$ on the fiber $\pi^{-1}(x)$ above x.

as Hol (c, ω). If we consider all possible loops based on x, the holonomies of connection ω around them will form a subgroup of G called the holonomy group of ω.

The holonomy of ω around a loop c is very important when we are considering the observables of YM theories. Accordingly to the gauge principle, these are gauge-invariant quantities [27]. Perhaps the simplest of such quantities can be extracted from Hol (c, ω) by taking its trace

$$W(c, \omega) = \text{Tr} \left[\text{Hol}(c, \omega) \right] . \tag{2.13}$$

This is known as a Wilson loop and it is common among physicists to express it in its local version

$$W(c, \omega) = \mathcal{P} \, \text{Tr} \left(e^{\int_c A} \right) , \tag{2.14}$$

where $\mathcal{P}$ is the path-ordering operator.

Other observables of YM theories can be constructed from F. For instance, the Pontryagin density

$$\mathcal{L}_{\text{Pontr}} = \text{Tr}(FF) \ , \tag{2.15}$$

the YM lagrangian density

$$\mathcal{L}_{\text{YM}} = \text{Tr}(F \star F) \ , \tag{2.16}$$

as well as observables with higher order in the curvature

$$O_{m,n} = \text{Tr}\left[(F \star F)\underbrace{\star(F \star F)\cdots\star(F \star F)}_{m\text{-times}}\star(FF)\underbrace{\star(FF)\cdots\star(FF)}_{n\text{-times}}\right] \ , \tag{2.17}$$

where $m, n \in \mathbb{N}$, $\star$ is the spacetime Hodge dual star operator[7]. In particular, the wedge product $\wedge$ was and will be omitted from now on.

It is important for us to notice that observables like $W(c, \omega)$ and $\mathcal{L}_{\text{Pontr}}$ carry global information about spacetime, i.e., they are topological and differential invariant, respectively, that, of course, do not dependent on a metric structure - notice that they lack $\star$: the metric dependent operation in the exterior algebra. Nonetheless, when we insert such topological observables in the path integral to obtain their vevs we, in general, metric-contaminate the result. The reason is that, in general, the path integral itself depends on a metric (the metric is usually present in the kinect terms, for example). In special cases though, it does not and the topological nature of these observables is preserved. These are exactly the cases for TQFT's, which will be introduced in chapter 3 of this thesis.

2.2.3 Moduli space

To finish this section, we will introduce some spaces which are of most importance in gauge theories. First, we will denote the set of all connection forms by $\mathcal{A}$. This is an affine, contractible manifold that thus has a trivial topology. The quotient space $\mathcal{A}/\mathcal{G}$, however, has not due to the usual nontrivial topology of $\mathcal{G}$. In fact $\mathcal{A}/\mathcal{G}$, usually denoted as $\mathcal{M}$ and refered to as the moduli space, may not even be a manifold. It is generally classified as an orbifold due to the presence of canonical singularities. This feature makes $\mathcal{M}$ very hard to work with,

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[7]The spacetime Hodge dual acts exclusively on a local coordinate basis dx^μ as

$$\star F = F_{\mu\nu} \star (dx^\mu dx^\nu) = F_{\mu\nu}\frac{1}{2}\epsilon^{\mu\nu}{}_{\alpha\beta}dx^\alpha dx^\beta \ .$$

where ϵ is the (metric dependent) permutation tensor.

though there are a few very special exceptions - the moduli space of instantons being one example.

The moduli space has much importance when one attempts to quantize gauge theories. Ideally, one should evaluate the path integral on $\mathcal{M}$ in order to avoid the so-called Gribov ambiguities [59, 50]. In practice, however, this is a very hard task (as already mentioned).

In non-Abelian gauge theories, in which $\mathcal{G}$ definitely has a nontrivial topology, the $\mathcal{G}$-bundle $\pi : \mathcal{A} \to \mathcal{M}$ is nontrivial, i.e., $\mathcal{A}$ cannot be globally seen as $\mathcal{M} \times \mathcal{G}$. In other words, the fibers of this bundle are twisted in such a messy way that it is actually impossible to cross each one of them only once, i.e., it is impossible to trace a global section in $\mathcal{A}$. Physically, this amounts on the impossibility to exclusively select only physically inequivalent gauge field configurations. In summary, it is not possible to completely fix the gauge field redundancies.

The persistent gauge ambiguities described above were discovered by V. N. Gribov in 1978 [59]. In his work, he argued that such configurations might play a very important role in the IR limit of YM theories. In particular, that a refined gauge fixing procedure might lead to a theory of confined quarks and gluons [59, 60, 50, 61, 62].

In this thesis, again, we will not worry about such issue. Since we will stay within the limits of perturbation theory, a local section on $\mathcal{A}$, i.e., a local gauge fixing will be enough. More precisely, we will only take into account field configurations that live within a very small region of $\mathcal{A}$, namely, the neighborhood of the classically trivial configuration A = 0. There, a local section is enough to pick only physically inequivalent configurations. We can then define a kind of perturbative version of the moduli space via the gauge fixing process and the Gribov obstruction is nonexistent.

2.3   BRST quantization

We start this section by asking the reader to consider M as the Euclidean 4-manifold $\mathbb{R}^4$ and G as any compact and simply-connected Lie group. These assumptions are very standard in the QFT framework. By considering M = $\mathbb{R}^4$ we avoid the discussion on how to evaluate singular momentum integrals in Minkowski spacetime by analytically extending them to the complex Euclidean space - where they become regular. Keep in mind though, that this is only a confortable position if we are going to employ perturbation theory (as we will).

The assumption of a compact and simply-connected Lie group, also called a semi-simple Lie group, ensures a positive-defined Killing metric. Thus, the kinect term of the YM gauge field is strictly positive and, as a consequence, the energy functional is bounded from below. In fewer words, this translates to the possibility of stable bound states.

2.3.1   Yang-Mills dynamics

Under the above conditions, the most general action functional that is polynomial in A and its first derivative, local, parity-preserving and power-counting renormalizable is

$$S_{YM}[A] = \int_{\mathbb{R}^4} \mathrm{Tr}\,(F \star F) \, , \tag{2.18}$$

famously known as the YM action. It, of course, is invariant under gauge transformations: henceforth denoted by its infinitesimal form

$$\delta A = -D\alpha \, , \tag{2.19}$$

where α is a $\mathfrak{g}$-valued 0-form on M, more commonly known as the infinitesimal gauge parameter. The extremization of the YM action (2.18) with respect to A leads to the YM field equation

$$D \star F = 0 \, , \tag{2.20}$$

that, together with Bianchi identity (2.10), encompass the on-shell dynamics of YM theories.

2.3.2   Pause for an instanton

We should pause for a moment to talk about a special class of solutions to the YM equations. These are self- and anti-self-dual solutions,

$$F^{\pm} = \pm \star F^{\pm} \, , \tag{2.21}$$

known as instantons. They represent classical field configurations that have a nontrivial topology. Indeed, a good example of when topology matters. In particular, they minimize the YM action (2.18) to

$$S_{YM}\left[A^{\pm}\right] = \pm \int \mathrm{Tr}\left(F^{\pm}F^{\pm}\right) \, , \tag{2.22}$$

which, as it was already mentioned, is a topological invariant[8]. In this context, (2.22) is also called the instanton number. It classifies the topology of a particular instanton field configuration.

Instantons are important because they play a vital role in YM theories. This is specially true due to their nonperturbative nature. For instance, in QCD, they amount for the huge degeneracy of perturbative QCD vacuum. This led to the idea of the more general θ-vacuum

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[8]Notice that the integration domain was omitted. This will be the case henceforth whenever it is clear, due to the context, that the integral is performed over spacetime $\mathbb{R}^4$.

structure. In fact, instantons can be understood, in this context, as quantum tunneling processes among these topologically inequivalent vacua [27]. Moreover, this kind of process can change the chirality of fermions. In this way, instantons can explain why the axial chiral symmetry $U(1)_A$ is anomalous, resulting in a hugh mass for the η' meson. This is the famous solution of the $U(1)$ problem given mainly by 't Hooft in the mid seventies [63, 45].

Instantons also break CP in the theory of strong interactions. This is yet to be observed in experiments and, for the moment, is regarded as an open problem in particle physics. Attempts to restore CP include the prediction of a new field, whose excitations are known as axions [64, 65]. In particular, axions may have a huge impact in the large scale structure of our Universe and they are also regarded as a strong candidate for cold dark matter component. For a review on axion cosmology see reference [66].

Most important here, instantons were the responsables for the development of TQFT's. In fact, it was their discovery that started major insertions of QFT into Differential Topology and vice-versa. Clearly, (2.22) represents this bridge: the absolute minimum of the YM action is a topological invariant: the 2nd Chern number. Indeed, it was exactly this fact that Witten explored, culminating in his 1988 papers [67–69]. We will leave the more detailed discussion to the next chapter. For now, let us get back to the understanding of BRST quantization.

2.3.3 Need of fixing gauge

The formal quantization of YM theories can be achieved by defining the path integral as

$$Z[J] = \int_{\mathcal{A}} \mathcal{D}A \, e^{-(S_{YM} + \int A \star J)} \tag{2.23}$$

where $\mathcal{D}A$ is the (ill-defined) functional measure on $\mathcal{A}$ and J is an external source. From this point on, the perturbative QFT framework dictates the steps to follow: i) to turn off the interactions and solve the free theory by obtaining the tree-level propagator of the gauge field; ii) turn on interactions and use perturbation theory to evaluate the quantum corrections these will impose upon this propagator, order by order in the loop expansion and; iii) abstract the results to a n-point Green function, i.e., to construct the Feynman's rules of YM theories. However, as soon as we try to fulfill step i) we stumble across a severe difficulty.

When interactions are turned off, the YM dynamics is given by the quadratic part of (2.18), a.k.a.,

$$S_{YM}^{(2)}[A] = \int \text{Tr} \, (A d \star dA) \; . \tag{2.24}$$

Its quite easy to see, specially because d squares to zero, that the functional form of the so-called wave operator $d \star d$, garantees the invariance of (2.24) under gauge transformations

$$\delta A = -d\alpha . \tag{2.25}$$

For this very same reason, we can see that this wave operator develops zero modes

$$d \star d\delta A = 0 . \tag{2.26}$$

It thus does not possess a well-defined inverse. In other words, we are unable to solve the free theory due to the presence of ambiguities in the gauge field.

2.3.4   Faddeev-Popov in a nutshell

A solution to the above impediment seems obvious: we have to find a way to fix the gauge. Gauge fixing non-Abelian gauge theories, however, is a very subtle procedure. If not done correctly, it might result in gauge anomaly, nonunitarity, nonrenormalizability, etc. The correct way was elucidaded in 1967 by physicists L. Faddeev and V. Popov [70].

The Fadeev-Popov (FP) procedure modifies the YM action, explicitly breaking its gauge invariance, by introducing a pair of unphysical scalar fields with fermionic statistics, known as the FP ghosts. In particular, the FP ghosts appear only as internal legs, i.e., virtual processes in the loop expansion. Their contributions exactly cancel the gauge anomaly, also preserving the unitarity and renormalizability of the theory.

The success of FP quantization in the perturbative framework contrasts itself to the very unnatural, ghostly recipe the procedure really is. After all, one might ask, i) why the introduction of such weird fields renders the theory so perfectly anomaly-free and renormalizable? ii) where these ghostly fields came from, in the first place?

2.3.5   BRST geometry

The answer to question i) only started clearing in 1975, by the works of physicists C. Becchi, A. Rouet, R. Stora [71] and, independently, I. Tyutin [72]. They understood that the FP procedure gauge fixes the action functional in such a way that a reminiscent global symmetry remained, the now-called Becchi-Rouet-Stora-Tyutin (BRST) symmetry. In particular, the BRST symmetry translates to a very strong set of Ward identities in the quantum gauge theory. For instance, its direct representative, the Slavnov-Taylor identity, forbids gauge anomalies and is pivotal to prove its renormalizability.

The answer to question ii), however, took longer. Indeed, it only cleared after the fiber bundle reformulation of non-Abelian gauge theory in the early eighties [56]. Only then, the meaning of the BRST operator s as well as the ghost field c, as natural geometrical objects within this mathematical setting, became apparent [73–76]. More recent developments can be found in [77, 78]. Indeed, the gauge field A and the ghost field c, as well as the exterior derivatire d and the BRST operator s, can be unified in a single geometrical object. Respectively,

$$\tilde{A} = A + c \, , \tag{2.27a}$$

$$\tilde{d} = d + s \, . \tag{2.27b}$$

On the principal $(G \times \mathcal{G})$-bundle $\pi : P \times \mathcal{A} \rightarrow M \times \mathcal{M}$, the object $\tilde{A}$ can be seen as a connection 1-form and $\tilde{d}$ as an exterior derivative on the base space $M \times \mathcal{M}$. Much like the previous case, they are actually the result of a pullback, via some section, of the connection form and exterior derivative defined on the fibers $P \times \mathcal{A}$. Notice that since $\tilde{d}$ squares to zero, the nilpotency of s is a natural consequence.

For practical purposes, one should notice that the exterior algebra over $M \times \mathcal{M}$ is graded by the form rank on M and the form rank on $\mathcal{M}$. The latter is known by physicists as the ghost number. For example, A is the $(1, 0)$ component of $\tilde{A}$, accordingly to the cross product $M \times \mathcal{M}$. This means that A is an 1-form on M, but a 0-form on $\mathcal{M}$ (zero ghost number). The sum of both ranks tells us that A has odd statistics. The ghost field, on the other hand, is the $(0, 1)$ component. It is a 0-form on M, but an 1-form on $\mathcal{M}$. In other words, it is a scalar field with ghost number 1. The sum of its ranks tell us that it also has odd statistics.

To $\tilde{A}$ we can associate a curvature

$$\tilde{F} = \tilde{d}\tilde{A} + \tilde{A}\tilde{A} \, , \tag{2.28}$$

which, as always, has to satisfy the corresponding Bianchi indentity

$$\tilde{D}\tilde{F} = 0 \, , \tag{2.29}$$

where $\tilde{D} \equiv \tilde{d} + \left[\tilde{A}, \quad \right]$ and $[\quad , \quad]$ now should be seen as Lie supercommutator, i.e., a commutator graded by the form rank on M and on $\mathcal{M}$[9]. By substituting (2.27) into (2.28) we can

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[9]The statistics of a field is given by the sum of its form and ghost ranks. If field X has statistics s_X and field Y has statistics s_Y, then $[X, Y] = XY - (-1)^{s_X s_Y}YX$. In physics, this is mostly known as a Lie superbracket or supercommutator due to its common use in supersymmetric field theories. Notice that it encompasses not only the tradicional notion of the commutator, but also of the anticommutator as well - depending on the statistics of the fields being evaluated.

write $\tilde{F}$ in a decomposed form

$$\tilde{F} = F + (sA + Dc) + (sc + cc) \ , \tag{2.30}$$

where F is its $(2, 0)$, $(sA + Dc)$ its $(1, 1)$ and $(sc + cc)$ its $(0, 2)$ component. The Bianchi identity (2.29) also decomposes, leading to

$$sF + DsA = 0 \ , \tag{2.31}$$

i.e., it is a constrait that forces the $(2, 1)$ component to be zero while the others vanish on their own. The BRST symmetry transformations of YM theories can then be obtained by the so-called horizontal condition

$$\tilde{F} = F \ , \tag{2.32}$$

i.e., that $\tilde{F}$ has nonvanishing components only in the direction of M. This choice makes component $(1, 1)$ reproduce the form of an infinitesimal gauge transformation, while component $(0, 2)$ gives c the meaning of a Maurer-Cartan form on $\mathcal{G}$

$$sA = -Dc \ , \tag{2.33a}$$

$$sc = -cc \ . \tag{2.33b}$$

On the other hand, Bianchi identity tell us the redundant fact that

$$sF = -\left[c, F\right] \ . \tag{2.34}$$

Notice that the horizontal condition in (2.32) represent a restriction of the BRST structure. Clearly, the BRST has a much more general setup then the particular case of YM theories. In the next chapter, we will investigate the case in which no components of $\tilde{F}$ vanish. Consequently, the BRST transformations will change and the resulting theory will drastically differ from YM.

2.3.6   BRST cohomology

The BRST quantization is much more than the differential geometric approach to the FP procedure. For instance, consider the cohomological groups[10] of M $\times$ $\mathcal{M}$ defined by the

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[10]Cohomological groups are topological invariants. They classify the particular topology of the manifold on which the correspondent nilpotent operator is defined. For completeness, let us make these definitions more explict. Consider d the nilpotent operator and $\Lambda^p(M)$ the space of all globally defined p-forms on M. The kernel

nilpotency of $\tilde{d}$,

$$\mathbb{H}^{p,q}\left(\mathrm{M}\times\mathcal{M},\tilde{d}\right)\sim\mathbb{H}^p\left(\mathrm{M},d\right)\oplus\mathbb{H}^q\left(\mathcal{M},s\right)\ ,\tag{2.36}$$

where $\sim$ is a group isomorphism, $\mathbb{H}^p\left(\mathrm{M},d\right)$ is the p-th cohomological group of M defined by the nilpotency of d and $\mathbb{H}^q\left(\mathcal{M},s\right)$ is the q-th cohomological group of $\mathcal{M}$ defined by the nilpotency of s. These groups have the information of all relevant gauge invariant objects of the theory: physical observables, anomalies, counterterms, etc. For instance, in the BRST formalism, gauge invariance translates to s-closeness. Observables that are s-exact are trivial because their vevs will vanish. This is due to the fact that $s\left|0\right\rangle=0$ in the absence of a spontaneous symmetry breaking. Therefore, all nontrivial physical observables are indeed contained within $\mathbb{H}^{(p,q)}\left(\mathrm{M}\times\mathcal{M},\tilde{d}\right)$. In practice, one can write

$$\tilde{d}O^{p,q}=0\ ,\tag{2.37}$$

where $O^{p,q}\in\Lambda^{p,q}\left(\mathrm{M}\times\mathcal{M}\right)$ are such observables. To find explict solutions one can decompose it in form and ghost rank, resulting in descent equations. Similar occurs for gauge anomalies, governed by $\mathbb{H}^{4,1}\left(\mathrm{M}\times\mathcal{M},\tilde{d}\right)$, and counterterms, governed by $\mathbb{H}^{4,0}\left(\mathrm{M}\times\mathcal{M},\tilde{d}\right)$. In particular, if $\mathrm{M}=\mathbb{R}^4$ then $\mathbb{H}^4\left(\mathrm{M},d\right)$ is trivial and the $(4,q)$-th cohomology is isomorphic to the q-th cohomology of the moduli space alone, i.e.,

$$\mathbb{H}^{4,q}\left(\mathrm{M}\times\mathcal{M},\tilde{d}\right)\sim\mathbb{H}^q\left(\mathcal{M},s\right)\ .\tag{2.38}$$

A very complete review on the BRST cohomology of gauge theories can be found in [79].

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of d,

$$\mathrm{Ker}(d)\equiv\left\{d\alpha=0,\ \forall\ \alpha\in\Lambda^p(\mathrm{M})\right\}\ ,$$

has a group structure and its elements are said to be d-closed forms. The image of d,

$$\mathrm{Im}(d)\equiv\left\{\alpha=d\beta,\ \forall\ \alpha\in\Lambda^p(\mathrm{M})\text{ and }\beta\in\Lambda^{(p-1)}(\mathrm{M})\right\}\ ,$$

also has a group structure and its elements are said to be d-exact forms. The p-th cohomological group of M is then the group of all global p-forms on M which are d-closed but not d-exact, i.e.,

$$\mathbb{H}^p\left(\mathrm{M},d\right)\equiv\frac{\mathrm{Ker}(d)}{\mathrm{Im}(d)}\ .\tag{2.35}$$

Whenever $\mathbb{H}^p\left(\mathrm{M},d\right)=\{\mathbb{1}\}$, i.e., it only contains the neutral element accordingly to the group operation, then it is said to be trivial. This means that M, at least accordingly to these topological invariants, has a trivial topology. In practice, this means that all d-closed p-forms on M are also d-exact.

2.3.7   BRST gauge fixing

We are now finally ready to put the BRST quantization into practice. Let us consider the Landau gauge condition

$$d \star A = 0 \ . \qquad \qquad \text{(Landau gauge)}$$

To implement this restriction on the YM action in a BRST-invariant fashion, we have to add to our theory a pair of fields, $\bar{c}$ and b, in a so-called BRST doublet structure

$$s\bar{c} = b \ , \qquad \qquad (2.39)$$

$$sb = 0 \ . \qquad \qquad (2.40)$$

Mathematically, $\bar{c}$ is $\mathfrak{g}$-valued 0-form on M with ghost number -1 and b is also a $\mathfrak{g}$-valued 0-form on M, but with ghost number 0. In physicists' terms, they are respectively known as the FP anti-ghost and the Lautrup-Nakanishi field[11]. The so-called doublet theorem ensures that fields introduced in such a way cannot belong to $\mathbb{H}^{p,q}\left(M \times \mathcal{M}, \tilde{d}\right)$ and thus do not alter the physical content of the theory [80].

The gauge fixing action for the Landau condition is then given by

$$\begin{aligned} S_{\text{gf}} &= s \int \text{Tr}\left(\bar{c}d \star A\right) \ , \\ &= \int \text{Tr}\left(bd \star A - \bar{c}d \star Dc\right) \ , \end{aligned} \qquad (2.41)$$

where, in the first line, s was applied to an integraded functional of fields A and $\bar{c}$, which was polynomial, power-counting renormalizable, local and had ghost number -1. The gauge fixing action, constructed in that way, is garanteed to be s-invariant since $s^2 = 0$.

Looking at (2.41) one can clearly see that the Landau condition is now derived as a field equation for b. In other words, the Landau gauge condition has been promoted from on-shell to off-shell configurations of the gauge field. Further, $-d \star D$ is clearly the FP operator which introduces not only the ghost-anti-ghost propagator but also the 3-vertex $-\bar{c}d \star [A, c]$. The latter is exactly the one responsable for a perfect cancellation of the unphysical contributions to the S-matrix coming from the naturally occuring 3-vertex $Ad \star [A, A]$. In return, we end up with a perfectly unitary quantum YM theory.

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[11]The watchful reader might be wondering, how can $\bar{c}$ have a negative form rank on $\mathcal{M}$? Well, it does not. It actually has an anti-BRST rank 1 on $\mathcal{M}$. The anti-BRST transformations, usually denoted as $\bar{s}$, are the twin sister of s. Actually, the geometrical setup of the BRST is better understood when one includes the anti-BRST. Nonetheless, this was only recently understood and involves a somewhat intricated double copying of $\mathcal{G}$. To avoid unnecessary complexity, this discussion was not included it in this thesis. Nonetheless, if the reader remains interested, the author suggests reference [78].

To end this section, we explictly write the path integral that defines such an unitary perturbative quantum YM theory, namely,

$$Z[J] = \int_{\mathcal{M}_{\text{pert}}} \mathcal{D}A\mathcal{D}c\mathcal{D}\bar{c}\mathcal{D}b \, e^{-\left(S_{\text{YM}}+S_{\text{gf}}+\int A \star J\right)} \, . \tag{2.42}$$

where $\mathcal{M}_{\text{pert}}$ is the perturbative, truncated version of $\mathcal{M}$.

2.4 Perturbative renormalizability

A consistent perturbative QFT not only has to be unitary but also has to renormalizable. The latter ultimately means that the theory has to be falsifiable: it has to have a finite set of free physical parameters that will, or will not, allow it to fit experimental data.

A perturbatively nonrenormalizable QFT, on the other hand, is not falsifiable. In order to extract finite predictions, one is forced to add to it new couplings order by order in perturbation theory. The resulting theory ends up with an arbitrarily large number of free parameters. Such immense freedom allows nonrenormalizable theories to literally fit any set of experimental data. They, therefore, should be dismissed as sensible physical theories[12].

In this section, it is our goal to show that quantum YM theories are physically sound QFTs in arbitrarily high energy scales. To achieve this, we have to prove that they are indeed renormalizable to all order in perturbation theory. The method of algebraic renormalizability [80, 61, 81] constitutes one of the most powerful tools one can choose to achieve this goal. The reason is twofold: i) it is independent of any regularization and/or renormalization scheme[13] and; ii) it is recursive by design, thus conclusions at 1-loop, for instance, can be extended to all order in perturbation theory.

The general idea basically consists of extending classical symmetries to quantum ones, order by order in perturbation theory, mainly to restrict the possible counterterms that might arise due to quantum corrections. In this paradigm, a QFT is renormalizable when it is stable, i.e., when all counterterms allowed by the quantum symmetries can be reabsorved back into

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[12]Perhaps this last phrase was a little bit too extreme. Nonrenormalizable theories are not always useless. Sometimes they can be understood as effective QFTs, i.e., UV incomplete theories that are not physically sound in arbitrarily high energy scales, but are still able to produce sensible results below some threshold energy scale. A particularly interesting example is pure General Relativity as a perturbative QFT in $d = 4$. It surely is a nonrenormalizable theory but, from an effective perspective, it can be applied, below Planck energy, to give a sensible 1-loop correction to the Newtonian potential [17].

[13]The philosophy behind the algebraic renormalizability method is to prove that a QFT can, or cannot, be renormalizable. Within this method, we do not actually renormalize the theory, i.e., we do not have to explicitly evaluate the divergent loop integrals. Therefore, there is no need to introduce any kind of regularization and/or renormalization scheme. This represents a great advantage since we avoid the danger of introducing artificial anomalies into the theory that might potentially cripple its renormalizability.

the original action functional by a finite number of redefinitions of parameters, fields and external sources. In this way, no indefinte number of free parameters has to introduced and the renormalized QFT ends up predictive and falsiable.

The algebraic renormalizability technique, of course, assumes a number of conditions for its results to be valid. In fact, it is only trustworthy when employed on local perturbative QFTs that are power-counting renormalizable and anomaly-free. We will assume these conditions to be true from now on.

2.4.1 Quantum action

One of the most important objects when renormalizability is at stake is the vertex functional $\Gamma\left[\langle A\rangle_J\right]$, where $\langle A\rangle_J \equiv \langle 0|\,A\,|0\rangle\,[J]$ is basically the vev of A in the presence of an external source J. Mathematically, the vertex functional is defined as the Legendre transformation

$$\Gamma\left[\langle A\rangle_J\right] \equiv \left(Z^c\,[J] - \int A \star J\right)\Bigg|_{A=\langle A\rangle_J}\,, \tag{2.43}$$

of the functional generator of connected Feynman graphs,

$$Z^c\,[J] \equiv -\ln\left(Z\,[J]\right)\,, \tag{2.44}$$

evaluated when A equals $\langle A\rangle_J$. Physically, the vertex functional generates one-particle irreducible Feynman graphs, i.e., only connected graphs, amputated from their external legs, and which remain connected after we cut any of their internal legs, if present. More loosely speaking, $\Gamma\left[\langle A\rangle_J\right]$ is a stripped down version of the path integral $Z\,[J]$, generating only the most elementary parts of the Feynman graphs: their vertexes and divergent loops. This is why the vertex functional is the preferred object to work with within this context.

Another very important fact about Γ is that it can be loop-expanded,

$$\Gamma = \Sigma^{(0)} + \epsilon\Sigma^{(1)} + \epsilon^2\Sigma^{(2)} + \cdots\,, \tag{2.45}$$

where ϵ is the small expansion parameter, and its tree-level term $\Sigma^{(0)}$ exactly equals the action functional of the classical theory,

$$\Sigma^{(0)} = S_{YM} + S_{gf}\,. \tag{2.46}$$

In other words, the vertex functional Γ as a whole can be seen as the classical action corrected by all the quantum effects. For this reason, it is more commonly called the quantum or effective action.

2.4.2   Quantum Action Principle

As already mentioned, one of our main objectives in the algebraic renormalizability technique is to extend classical (tree-level) symmetries to quantum ones. In other words, we would like to extended the symmetries of $\Sigma^{(0)}$ to symmetries of Γ.

The Quantum Action Principle (QAP) [80, 61] is particularly useful for this purpose. This principle is based on a series of formal results in QFT, independent of any regularization and/or renormalization scheme [82–87]. It can be seen as the quantum analog of Hamilton's principle, in the sense that it establishes a well-defined version of the quantum equations of motion, known as the Dyson-Schwinger equations.

On the other hand, the QAP can also be interpreted as a recipe on how classical operators can be extended to quantum ones. For instance, in the Γ representation, the QAP states that

$$\frac{\delta\Gamma}{\delta A} = \Delta \cdot \Gamma \, , \tag{2.47}$$

i.e., it tell us that the action of $\delta/\delta A$ on Γ is exported to the quantum theory as an insertion[14] of a local field operator Δ into Γ itself. In particular, $\Delta \cdot \Gamma$, at lowest order, has to coincide with its classical counterpart,

$$\Delta \cdot \Gamma = \frac{\delta\Sigma^{(0)}}{\delta A} + O\left(\epsilon\right). \tag{2.48}$$

In other words, the QAP states that a classical operator enters the quantum realm by suffering all possible quantum corrections since it ends up literally inserted into the correlation functions evaluated from $\Delta \cdot \Gamma$.

This is precisely what we need to properly extend classical symmetries, written in their functional form, to quantum ones. We will do this in the next section. Before, however, we must deal with a peculiarity when nonlinear symmetries are present.

Accordingly to the QAP, nonlinear symmetry operators, such as the BRST, when acting upon Γ, end up being inserted into correlation functions as polynomials of the fields evaluated

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[14]The notation $\Delta \cdot Z$ stands for

$$\Delta \cdot Z = \int_{\mathcal{M}_{\text{pert}}} \mathcal{D}A\mathcal{D}c\mathcal{D}\bar{c}\mathcal{D}b\,\Delta e^{-\left(S_{\text{YM}}+S_{\text{gf}}+\int A\star J\right)} \, ,$$

i.e., Δ is literally inserted into the correlation functions evaluated from $\Delta \cdot Z\,[J]$.

at the same point of spacetime. In other words, as new vertexes in the Feynman graphs that carry divergencies not accounted for by the path integral (2.42).

In order to remedy the above situation, we have to explicitly add these nonlinearities to the classical action. This must be done in a BRST invariant fashion, of course. For the case of the BRST nonlinearities themselves, we introduce two pairs of external sources in a BRST doublet structure

$$s\tau = \Omega \,, \quad s\Omega = 0 \,, \tag{2.49a}$$

$$sE = L \,, \quad sL = 0 \,. \tag{2.49b}$$

A summary of all the fields introduced in this chapter, together with their gradings, can now be found in table 2.1. We thus define the so-called external action as

$$S_{\text{ext}} = s \int \text{Tr} \left(\tau Dc + Ecc \right) \,,$$

$$= \int \text{Tr} \left(\Omega Dc + Lcc \right) \,. \tag{2.50}$$

The full classical action of interest is now

$$\Sigma \equiv S_{\text{YM}} + S_{\text{gf}} + S_{\text{ext}} \,. \tag{2.51}$$

The new path integral, now accounting for all divergent objects, is given by

$$Z[J] = \int_{\mathcal{M}_{\text{pert}}} \mathcal{D}A\mathcal{D}c\mathcal{D}\bar{c}\mathcal{D}b \, e^{-\left(\Sigma + \int A \star J\right)} \,, \tag{2.52}$$

and the corresponding quantum action Γ now expands as

$$\Gamma = \Sigma + \epsilon \Sigma^{(1)} + \epsilon^2 \Sigma^{(2)} + \cdots \,. \tag{2.53}$$

Table 2.1 The fields of quantum YM theory and their grading as differential forms on M and on $\mathcal{M}$, respectively.

Fields	A	c	$\bar{c}$	b	τ	Ω	E	L
M	1	0	0	0	3	3	4	4
$\mathcal{M}$	0	1	-1	0	-2	-1	-3	-2

2.4.3   Ward identities

We will now start the process of exporting symmetries of Σ to symmetries of Γ using, of course, the QAP. We will, at all times, assume that no anomalies are present. Let us begin with the gauge fixing equation

$$\frac{\delta \Sigma}{\delta b} = d \star A \, , \tag{2.54}$$

which can be seen as a local symmetry of the classical action Σ broken by a linear term $d \star A$ in the field A. Accordingly to the QAP, this "symmetry" is exported to Γ as

$$\frac{\delta \Gamma}{\delta b} = d \star A + \mathcal{O}\left(\epsilon\right) \, , \tag{2.55}$$

which, as we are assuming no anomalies are present, resumes to

$$\frac{\delta \Gamma}{\delta b} = d \star A \, . \tag{2.56}$$

Though the gauge fixing equation does not represent an actual symmetry of either Σ or Γ, we will see in the next section that it will be a symmetry for the counterterms. Thus its importance.

Another local symmetry is the FP antighost equation

$$\mathcal{G}_{\bar{c}}\left(\Sigma\right) = 0 \, , \tag{2.57}$$

where

$$\mathcal{G}_{\bar{c}} \equiv \left(\frac{\delta}{\delta \bar{c}} + d \star \frac{\delta}{\delta \Omega} \right) \, . \tag{2.58}$$

At this time, the symmetry is exact. It translates to

$$\mathcal{G}_{\bar{c}}\left(\Gamma\right) = 0 \, , \tag{2.59}$$

if, again, no anomalies are present (this conditional has been repeated too many times and will be ommited from now on).

Now we display the global (integrated) symmetries. We start with the BRST symmetry

$$\mathcal{S}\left(\Sigma\right) = 0 \, , \tag{2.60}$$

where

$$\mathcal{S} \equiv \int \mathrm{Tr} \left(-\frac{\delta}{\delta \Omega} \frac{\delta}{\delta A} - \frac{\delta}{\delta L} \frac{\delta}{\delta c} + b \frac{\delta}{\delta \bar{c}} \right) \, , \tag{2.61}$$

is the so-called Slavnov-Taylor operator, which is nothing but the BRST operator s written in a functional form. It translates to the quantum action as

$$\mathcal{S}(\Gamma) = 0 \,, \tag{2.62}$$

which is the well-known Slavnov-Taylor identity.

All the symmetries displayed above are algebraically related to each other. Consider, for instance, a functional Θ of the fields and external sources. It is then true that

$$\frac{\delta}{\delta b} \mathcal{S}(\Theta) - \mathcal{S}_{\Theta} \left(\frac{\delta \Theta}{\delta b} - d \star A \right) = \mathcal{G}_{\bar{c}}(\Theta) \,, \tag{2.63}$$

where

$$\mathcal{S}_{\Theta} \equiv \int \mathrm{Tr} \left(-\frac{\delta \Theta}{\delta \Omega} \frac{\delta}{\delta A} - \frac{\delta \Theta}{\delta A} \frac{\delta}{\delta \Omega} - \frac{\delta \Theta}{\delta L} \frac{\delta}{\delta c} - \frac{\delta \Theta}{\delta c} \frac{\delta}{\delta L} + b \frac{\delta}{\delta \bar{c}} \right) \tag{2.64}$$

is the so-called linearized Slavnov-Taylor operator. In particular, whenever $\mathcal{S}(\Theta) = 0$, then $\mathcal{S}_{\Theta}\mathcal{S}_{\Theta} = 0$ and thus it defines a cohomology. This operator is specially important in the renormalizability of YM theories. As we will see in the next section, due to the perturbative and recursive nature of the algebraic renormalizability technique, it is the $(4, 0)$-th cohomological group of $\mathcal{S}_{\Gamma}$, not of the full nonlinear BRST operator s, that will be responsible to effectively restrict the possible counterterms. This, however, does not contradict what has been said in section 2.3.6. The reason is that, by a redefition of fields, one can show that the $(4, 0)$-th cohomological group of s and the $(4, 0)$-th cohomological group of $\mathcal{S}_{\Gamma}$ are actually one isomorphic to the other.

The last relevant symmetry is the linearly broken FP ghost equation

$$\mathcal{G}_c(\Sigma) = \Delta_c \,, \tag{2.65}$$

where

$$\mathcal{G}_c \equiv \int \left(\frac{\delta}{\delta c} - \left[\bar{c}, \frac{\delta}{\delta b} \right] \right) \tag{2.66}$$

and

$$\Delta_c \equiv \int \left([A, \Omega] + [c, L] \right) \,. \tag{2.67}$$

As expected, it extends to the quantum theory as

$$\mathcal{G}_c(\Gamma) = \Delta_c \,. \tag{2.68}$$

From now on, we will collectively call equations (2.56, 2.59, 2.62, 2.68) as the Ward identities of quantum YM theories in the Landau gauge.

2.4.4 Counterterms

For the Ward identities to hold true, they must be satisfied order by order in the loop expansion. Indeed, at loop orders higher than the tree-level, one explicitly finds that

$$\frac{\delta \Sigma^{(n)}}{\delta b} = 0 \,, \tag{2.69a}$$

$$\mathcal{G}_{\bar{c}} \left(\Sigma^{(n)} \right) = 0 \,, \tag{2.69b}$$

$$\mathcal{S}_{\Gamma^{(n-1)}} \left(\Sigma^{(n)} \right) = 0 \,, \tag{2.69c}$$

$$\mathcal{G}_{c} \left(\Sigma^{(n)} \right) = 0 \,, \tag{2.69d}$$

$\forall \, n \in \mathbb{N}^*$, where

$$\Gamma^{(n)} \equiv \Sigma + \epsilon \Sigma^{(1)} + \cdots + \epsilon^n \Sigma^{(n)} \tag{2.70}$$

is the quantum action truncated at order ϵ^n. As antecipated in the previous sections, the linear breakings are gone, i.e., they occur and remain at a classical level[15]. Here the recursive nature of the algebraic renormalizability method also becomes apparent. Since (2.69) are valid at any n, conclusions extrated from them at a particular loop order can be extended to any loop order, and thus to all orders in perturbation theory. For convenience, the usual choice is to work at $n = 1$. Also, it is usual to denote $\Sigma^{(1)}$ as Σ^{ct}.

Our task is then to find the most general Σ^{ct} that is compatible with the Ward identities (2.69). In particular, equation (2.69c) for $n = 1$ states that Σ^{ct} belongs to the kernel of $\mathcal{S}_{\Sigma}$. Most generally, this means that it has a piece that belongs to the $(4, 0)$-th cohomological group of $\mathcal{S}_{\Sigma}$ and another piece that belongs to the image of $\mathcal{S}_{\Sigma}$. Explicitly,

$$\Sigma^{\mathrm{ct}} = \Delta^0 + \mathcal{S}_{\Sigma} \Delta^{-1} \,, \tag{2.71}$$

where Δ^0 must be an integrated element of $\mathbb{H}^{(4,0)} \left(\mathrm{M} \times \mathcal{M}, \tilde{d} \right)$ and Δ^{-1} must also be a integraded element of $\mathbb{H}^{(4,-1)} \left(\mathrm{M} \times \mathcal{M}, \tilde{d} \right)$. Here the isomorphism among the cohomological groups of s and $\mathcal{S}_{\Sigma}$ was used.

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[15]The same cannot be said about nonlinear breakings. These propagate beyond the tree-level and, therefore, will not translate to perturbative symmetries of the quantum theory.

The most general Δ^0 can only be the YM action itself,

$$\Delta^0 = a_0 S_{YM} \, , \tag{2.72}$$

while the most general Δ^{-1} is

$$\Delta^{-1} = \int \mathrm{Tr} \left(a_1 \bar{c} d \star A + a_2 \bar{c} \star b + a_3 \tau Dc + a_4 Ecc + a_5 A\Omega + a_6 \bar{c}\bar{c} \star c + a_7 cL \right) . \tag{2.73}$$

The parameters a_i are arbitrary.

The application of $\mathcal{S}_\Sigma$ on Δ^{-1} kills coupling a_3 and a_4. What is also forbidden to appear in Σ^{ct} is b, due to Ward identity (2.69a). As a result, $a_5 = -a_1$ and $a_6 = a_2 = 0$. Further, Ward identity (2.69d) restricts the possible couplings of c. It couples only with d-exact 4-form (with ghost number -1). As a consequence, $a_7 = 0$. The final result is

$$\Sigma^{ct} = a_0 S_{YM} - a_1 \int \mathrm{Tr} \left[2AD \star F + c \left(d \star d\bar{c} - d\Omega \right) \right] . \tag{2.74}$$

This is the most general counterterm allowed by the Ward identities (2.69) of the quantum YM theories. It was evaluated as an 1-loop correction to classical action but, since the method is recursive, it is actually the most general n-loop correction of $\Gamma^{(n-1)}$.

2.4.5 Quantum stability

Finally, one can show that Σ^{ct} can be absorbed by Σ by a redefinition

$$\Phi_0 \equiv z_\Phi \Phi \, , \tag{2.75a}$$

$$g_0 \equiv z_g g \, , \tag{2.75b}$$

$$\mathcal{J}_0 \equiv z_{\mathcal{J}} \mathcal{J} \, , \tag{2.75c}$$

of fields $\Phi \in \{A, c, \bar{c}, b\}$, parameter g and external sources $\mathcal{J} \in \{\Omega, L\}$. In other words, that

$$\Sigma \left[\Phi, g, \mathcal{J} \right] + \epsilon \Sigma^{ct} \left[\Phi, g, \mathcal{J} \right] = \Sigma \left[\Phi_0, g_0, \mathcal{J}_0 \right] \, . \tag{2.76}$$

To clarify, g is the coupling parameter of the self-interacting YM field A. Until now, we have omitted it from the equations. Nevertheless, it can be easily introduced back by considering the curvature of A as $F = dA + gAA$.

The nontrivial z-factors are

$$z_g = 1 - \epsilon \frac{a_0}{2} \,, \tag{2.77a}$$

$$z_c = 1 + \epsilon \frac{a_1}{2} \,, \tag{2.77b}$$

$$z_A = z_g^{-1} z_c^{-2} \,, \tag{2.77c}$$

$$z_\Omega = z_{\bar{c}} = z_c \,. \tag{2.77d}$$

Clearly, there are only two independent renormalizations. In particular, the only physical renormalization is the renormalization of g. This is due to the fact that g renormalizes with the nontrivial counterterm Δ^0.

We now have firm grounds to state that quantum YM is stable under quantum corrections and, therefore, is predictive and falsifiable in arbitrarily high energies. Hopefully, this review was enough for the reader to understand the algebraic renormalizability technique and its importance in determining the quantum consistency of a given QFT. If not, the reader should follow the references cited along this chapter. From now on, the algebraic renormalizability technique will be used in a much less descriptive manner.

Chapter 3

Topological Yang-Mills theories

3.1 Introduction

The interaction between Topology and Physics started in relatively recent times[1]. One of the first conscient uses of topological concepts in physical models appeared in 1955, in a paper entitled "Geons" by J. A. Wheeler [89].

In his work, Wheeler resurrected a program started by A. Einstein and N. Rosen twenty year before [90]. The idea was to remove undesirable singularities not only from General Relativity but from all classical field theories. To this aim, Wheeler considered, for instance, the possibility of a nontrivial spacetime topology being the agent responsable for electric monopoles, see figure 3.1. The concept of an electric field with singularities would then be replaced by an everywhere smooth field on a multiply connected space. By the way, this was the first time a sketch of the modern notion of a wormhole appeared in literature.

On the other hand, topological objects were present in many physical models developed before Wheeler's work. Their particular nature, however, was only realized much later on. A typical example is the magnetic monopole, proposed by P. A. M. Dirac in the thirties [91]. Only in mid seventies, however, it was understood as a natural solution of Maxwell's equations over a sphere [92].

There is no doubt that Topology joined mainstream Physics after the rise of gauge theory. Physicists learned that gauge theories had a formulation in terms of fiber bundles, as reviewed in the last chapter. Moreoever, G. 't Hooft and A. M. Polyakov, independently, discovered nonsingular topological solutions of the YM equations, now known as 't Hooft-Polyakov monopoles [93, 94].

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[1]For a much more detailed and accurated historical account on the interplay between Topology and Physics, see [88].
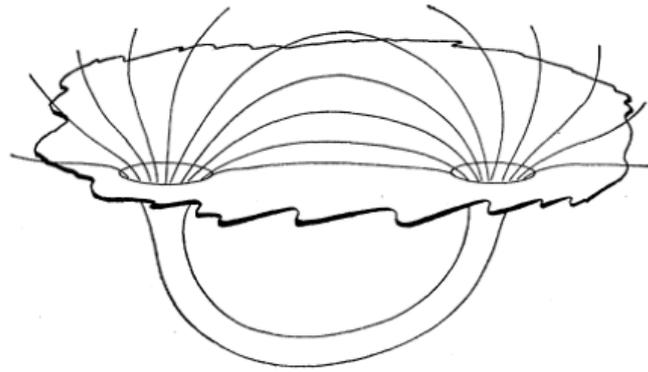
FIG. 7. Schematic representation of lines of force in a doubly-connected space. In the upper continuum the lines of force behave much as if the tunnel mouths were the seats of equal and opposite charges.

Figure 3.1 Extracted from "Geons" by J. A. Wheeler [89].

At about the same time, A. A. Belavin *et al* discovered another class of topological solutions to the YM equations in Euclidean spacetime: instantons [95]. As discussed in the last chapter, they are of major importance in the nonperturbative regime of YM theory and also in Differential Topology.

Due to instantons, mathematicians started appreciating the connection between Physics and Mathematics. The works of R. S. Ward and M. F. Atiyah were crucial in this direction. They were able to show that information contained in these (anti-)self-dual solutions of the YM equations could be enconded in certain vector bundle structures [96, 97]. This made the hard analytical problem of solving the YM equations into a topological, more accessible one. It would not take long for a large number of mathematicians and physicists to be working on very closely related problems.

The full extent in which QFT overlaps Topology, and vice-versa, was only unveiled in the eighties, after the works of S. K. Donaldson [98–102]. In them he used the YM instantons to make important breakthroughs in the classification of smooth 4-manifolds. More specifically, he realized that the instanton moduli space $\mathcal{M}_{\text{ins}}$ and the spacetime 4-manifold M had a very close relation, namely, it was possible to express differential invariants[2] of M as integral of differential forms on $\mathcal{M}_{\text{ins}}$. These are the famous Donaldson invariants.

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[2]While topological invariants classify the topological structure of a topological space, differential invariants classify the differential structures, a.k.a., C^∞-atlases or smooth structure, that we can endow a topological space with. So, two smooth manifolds can be homeomorphic to (same topology) but not diffeomorphic to (different smooth structure) each other. This is not so relevant in $d < 4$, since in these dimensions the smooth structure is unique. But the situation drastically changes for $d \geqslant 4$ and is particularly difficult in $d = 4$. For instance, there are an uncoutable number of "exotic" $\mathbb{R}^4$, i.e., manifolds homeomorphic to the Euclidean space, $\mathbb{R}^4$, but not diffeomorphic to it.

The next great advance was due to mathematical-physicist E. Witten, which himself described the historical scenario in [67]:

> *"In a lecture at the Hermann Weyl Symposium last year, Michael Atiyah proposed two problems for quantum field theorists. The first was to give physical interpretation to Donaldson theory. The second problem was to find an intrisically three dimensional definition of the Jones polynomials of knot theory."*

Witten was able to solve both problems in his 1988 papers [67, 69, 68]. He demonstrated that both Donaldson and Jones invariants, as well as others invariants, could be obtained from a special kind of quantum gauge field theories, now known as topological quantum field theories.

These QFTs can be very generally defined as the ones whose vevs of physical observables necessarily satisfy equation (3.1),

$$\frac{\delta \langle O \rangle}{\delta g_{\mu\nu}} = 0 \ , \tag{3.1}$$

where

$$\langle O \rangle \equiv O \cdot Z[J] \ . \tag{3.2}$$

In other words, they are are metric-independent quantities: topological or differential invariants classifying global features of the spacetime manifold.

The absence of a metric indicates that there is no usual local dynamics: observables are not measured by local rods and clocks. Moreover, the Hilbert space of TQFTs is specially simple: usually finite dimensional, containing only degenerated vacuum states. TQFTs, however, have a nontrivial global dynamics, consisting of tunneling processes among these topologically inequivalent vacua. This is the main physical content of such theories.

Examples of TQFTs are: i) $d = 2$ topological sigma models related to Gromov-Witten invariants classifying holomorphic maps between a Riemannian surface and a target space [68]; ii) $d = 3$ Chern-Simons theory, related to link and knot invariants (Jones polynomials) of submanifolds embedded in $\mathbb{R}^3$ [67]; iii) $d = 4$ topological Yang-Mills theory (TYM), related to Donaldon invariants of smooth 4-manifolds [69] and finally; iv) gravity itself, in lower spacetime dimensions: two-dimensional gravity can be associated to Mumford-Morita-Miller invariants [103] and three-dimensional gravity was shown to be perturbatively equivalent to a Chern-Simons theory in [104].

In this thesis, we are interested in four spacetime dimensions. We will thus concentrate our discussion on the four-dimensional TYM theory, whose partition function, as already said, gives an exact integral representation to Donadson invariants.

There are actually many equivalent ways to construct such a theory. Due to its intricated construction, we will only briefly comment on Witten's original approach. A much clear and easy-to-understand formulation was given only two months after by L. Baulieu and I. M. Singer in [76]. Thus, for pedagogical reason we will stick to the latter. I should also add that there is a more mathematical and, possibly, the most geometrical approach of all, which is the construction via the Mathai-Quillen formalism [105].

3.2 Witten approach in a nutshell

Witten first formulated a TQFT for the Donaldson invariants as a "twisted" version[3] of $\mathcal{N} = 2$ Super-YM theory [69]. Its dynamics is defined by path integral

$$Z[J] = \int_{\mathcal{A}} \mathcal{D}\Phi \, e^{\left(S_{\text{twisted}}[\Phi] + \int A \star J\right)} , \tag{3.3}$$

where

$$S_{\text{twisted}}[\Phi] = \int \text{Tr} \left(F \star F + FF + \phi D \star D\lambda + \eta D \star \psi + D\psi \star \chi + \phi[\chi, \star\chi] + \lambda[\psi, \star\psi] + \right.$$
$$\left. + \phi[\eta, \eta] \star \mathbb{1} + [\phi, \lambda]^2 \star \mathbb{1}\right) , \tag{3.4}$$

and $\Phi \in \{A, \phi, \lambda, \psi, \eta, \chi\}$.

We will not try to fully understand Witten's intricated construction and, if the reader is interested in a review, he or she may check references [106, 107]. Rather, we will focus on two key aspects of (3.3). The first is that it satisfies

$$\frac{\delta Z}{\delta g_{\mu\nu}} = 0 , \tag{3.5}$$

which means that the path integral itself is a topological invariant. The second is that

$$\frac{\delta Z}{\delta g} = 0 , \tag{3.6}$$

where g is the same g as in $F = dA + gAA$. These two properties were the key ingredients in Witten's reasoning.

The first one is a necessary condition for the vevs of observables to be topological, i.e., for (3.1) to hold[4]. The second one means that the theory is insensitive to the values of the

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[3]The "twist" is a linear map that associates spinorial degrees of freedom to vectorial ones [?]. Further, the fermionic generator of the sypersymmetric algebra behaves much like a BRST operator of the twisted theory.

[4]A metric dependent path integral contaminates with a metric every vev evaluated from it.

coupling parameter g. In other words, the weekly and strongly coupled regime are the same. Witten used this latter fact to argue that the semi-classical approximation is actually exact. Since this regime is dominated by the classical minima, a.k.a., instantons, the path integral over $\mathcal{A}$ could then be replaced, without loss, by a path integral over $\mathcal{M}_{\text{ins}}$. This is how Witten could reproduce Donaldson invariants.

3.3   Baulieu-Singer approach

At the concluding section of his intricated construction, Witten states

> *"The fermionic symmetry that we have used is very reminiscent of BRST symmetry. Its use is quite similar to the use of BRST symmetry in string theory. So it is natural to think that in a suitable framework, this symmetry arises upon BRST gauge fixing of an underlying gauge invariant theory."*

Witten was not able, however, to propose such underlying gauge theory, he continues

> *"One of the real mysteries is how to exhibit a manifestly generally covariant theory whose BRST gauge fixing (at least in some approximation) gives "topological quantum field theory" we have considered."*

The general covariance Witten refers to is related to equation (3.1), i.e., a gauge theory whose observables are metric-independent.

Witten suspicion was confirmed only two month after, by the work of L. Baulieu and I. M. Singer [76]. The Baulieu-Singer approach greatly simplified Witten's original construction. It is much more geometrical in nature and gave valuable insights on the meaning of the BRST operator.

3.3.1   BRST structure

Let us consider again the principal $(G \times \mathcal{G})$-bundle $\pi : P \times \mathcal{A} \to M \times \mathcal{M}$ in which the gauge field A and the ghost field c as well as the exterior derivative d and the BRST operator s are unified in a single mathematical object, respectively,

$$\tilde{A} = A + c \, , \tag{3.7a}$$

$$\tilde{d} = d + s \, . \tag{3.7b}$$

As we saw, to $\tilde{A}$ we can associate the curvature

$$\tilde{F} = \tilde{d}\tilde{A} + \tilde{A}\tilde{A} ,$$
$$= F + (sA + Dc) + (sc + cc) . \tag{3.8}$$

The YM BRST can be obtained from (3.8) by imposing horizontal condition $\tilde{F} = F$. This BRST, however, is no good for TQFTs since it allows for observables to be metric-dependent. It turns out that the BRST Witten was looking for, and that was found by Baulieu and Singer, is the unrestricted, general one, that emerges naturally from the fiber bundle geometry and provides $\tilde{F}$ all of its components. In other words, the one that do not obey - at all - the horizontal condition

$$\tilde{F} = F + \psi + \phi , \tag{3.9}$$

where $\psi \equiv sA + Dc$ is the $(1, 1)$ and $\phi \equiv sc + cc$ the $(0, 2)$ component of $\tilde{F}$. The full set of transformations can then be written as

$$sA = -Dc + \psi , \tag{3.10a}$$
$$sc = -cc + \phi , \tag{3.10b}$$
$$s\psi = -D\phi - [c, \psi] , \tag{3.10c}$$
$$s\phi = -[c, \phi] , \tag{3.10d}$$

where ψ as well as ϕ also have to transform in order to maintain the nilpotency of s. Further, Bianchi identity $\tilde{D}\tilde{F} = 0$, tell us the redudant fact that

$$sF = -D\psi - [c, F] . \tag{3.11}$$

What is then, the gauge theory behind Witten's twisted construction? Well, (3.10a) is the BRST gauge fixed version of the gauge transformation

$$\delta A = -D\alpha + \beta , \tag{3.12}$$

where α is the usual infinitesimal gauge parameter and β is a novel one associated with a general transformation of the gauge field A. This is clearly a much stronger symmetry than the traditional gauge one. The only action functional that remains invariant under it is the one which is a number or, better, a topological invariant - if α and β belong to the same

topological sector as A. For instance, the integral of the Pontryagin density

$$S_{TYM} = \int \text{Tr}\,(FF) \,, \tag{3.13}$$

giving the 2nd Chern class or, if the gauge group is (pseudo-)orthogonal, the integral of the Gauss-Bonnet density

$$S_{TYM} = \int \text{Tr}\,(FF^\star) \,, \tag{3.14}$$

where $F^\star$ is the Lie dual[5] of F, giving Euler invariant.

A very clear picture now starts to emerge. Full BRST invariance under (3.10) forbids the presence of the metric $g_{\mu\nu}$, via the spacetime Hodge dual $\star$, in the starting action. Then, the YM lagrangian $\text{Tr}\,(F \star F)$ must be replaced by $\text{Tr}\,(FF)$ or $\text{Tr}\,(FF^\star)$. The metric will end up being inserted into the theory, due to the necessity of a gauge fixing, but, since this is done in a s-exact way, the observables will not be contaminated, i.e., they will remain topological. This is why the name of Topological YM theory is justified.

3.3.2 Observables

In this approach, the Donaldson invariants can be understood as Chern classes classifying the inequivalent vector bundle structures one can construct over $M \times \mathcal{M}$,

$$O_n = \text{Tr}\,\big(\underbrace{\tilde{F} \cdots \tilde{F}}_{n\text{-times}}\big) \,. \tag{3.15}$$

where $n \in \mathbb{N}$ and O_n is the n-th Chern class. In particular,

$$O_2 = 2\,\text{Tr}\left[\frac{1}{2}FF + \psi F + \left(\phi F + \frac{1}{2}\psi\psi\right) + \psi\phi + \frac{1}{2}\phi\phi\right] \tag{3.16}$$

is exactly the one evaluated by Witten in [69]. For more details on Donalson's work, the relation of Chern classes and the smooth structure of spacetime, the author highly recommends reference [108].

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[5]The Lie dual $\star$ acts exclusively on the Lie algebra generators

$$F^\star = F^{ab}\sigma^\star_{ab} = F^{ab}\frac{1}{2}\epsilon_{ab}{}^{cd}\sigma_{cd} \,.$$

It is a Hogde dual on G, a "color" Hodge dual, and should not be confused with the usual Hodge dual $\star$ on spacetime M, introduced in chapter 2.

3.4   Equivalence of approaches

Notice in (3.10) and (3.11) that not only A transformas as a gauge field, but also ψ and F as well. We thus need three gauge fixing conditions for the BRST quantization of S_{TYM}: one for A, one for ψ and one for F. Baulieu and Singer showed that the DW action (3.4) can be obtained (plus some extra ghostly interactions) from the BRST gauge fixing of S_{TYM} using the nonlinear constraits

$$d \star A = -\xi_1 \star b \,, \tag{3.17a}$$

$$D \star \psi = 0 \,, \tag{3.17b}$$

$$F \pm \star F = -\xi_2 \star B \,, \tag{3.17c}$$

where ξ_i are gauge fixing parameters and B is the analogous of b for the gauge condition of F. This scenario is exactly the one Witten suspected but could not show.

3.5   Perturbative renormalizability

We will now employ the algebraic renormalizability technique to show that the TYM theory renormalizable to all orders in perturation theory and thus represents a consistent QFT to arbitrarily high energies.

   We will not use the nonlinear gauge given in (3.17). Rather, we will use a simpler, linear one, given by

$$d \star A = 0 \,, \tag{3.18a}$$

$$F \pm \star F = 0 \,, \tag{3.18b}$$

$$d \star \psi = 0 \,. \tag{3.18c}$$

which we will call the (anti-)self-dual Landau [(A)SDL] gauge. To implement it, we will introduce three pairs of BRST doublets

$$s\bar{c} = b \,, \quad sb = 0 \,, \tag{3.19a}$$

$$s\bar{\chi} = B \,, \quad sB = 0 \,, \tag{3.19b}$$

$$s\bar{\phi} = \bar{\eta} \,. \quad s\bar{\eta} = 0 \,. \tag{3.19c}$$

The gauge fixing action is then given by

$$
S_{gf} = s \int \text{Tr} \left[\bar{c}d \star A + \bar{\chi} \left(F \pm \star F \right) + \bar{\phi} d \star \psi \right] ,
$$
$$
= \int \text{Tr} \left\{ bd \star A - \bar{c} d \star Dc + \left(\bar{c} + \bar{\eta} + \left[c, \bar{\phi} \right] \right) d \star \psi + \bar{\phi} d \star D\phi +
$$
$$
+ \, dc \left[\star \psi, \bar{\phi} \right] + \left(B + \left[\bar{\chi}, c \right] \right) \left(F \pm \star F \right) + \bar{\chi} \left(D \pm \star D \right) \psi \right\} . \tag{3.20}
$$

As we will see, our quantum TYM enjoys a very strong set of Ward identities. Some of them are nonlinear, such as the Slavnov-Taylor identity, and, as discussed in the last chapter, these nonlinearities must be explicitly included in the action. To this aim, we introduce three more pairs of BRST dublets, namely,

$$
s\tau = \Omega , \quad s\Omega = 0 , \tag{3.21a}
$$
$$
sE = L , \quad sL = 0 , \tag{3.21b}
$$
$$
s\lambda = K , \quad sK = 0 . \tag{3.21c}
$$

The external action is then

$$
S_{ext} = s \int \text{Tr} \left(\tau Dc + Ecc + \lambda \left[c, \bar{\chi} \right] \right) ,
$$
$$
= \int \text{Tr} \left\{ \Omega Dc + \tau \left(D\phi + \left[c, \psi \right] \right) + Lcc + E \left[c, \phi \right] + K \left[c, \bar{\chi} \right] + \lambda \left(\left[c, B \right] + \right.
$$
$$
+ \left. \left[cc - \phi, \bar{\chi} \right] \right) \right\} . \tag{3.22}
$$

Finally, the total action of interest is given by

$$
\Sigma = S_{TYM} + S_{gf} + S_{ext} \tag{3.23}
$$

and the gradings of all fields can be found in table 3.1.

Table 3.1 Gradings of the fields as differential forms on M and on $\mathcal{M}$.

Fields	A	c	ψ	ϕ	$\bar{c}$	b	$\bar{\chi}$	B	$\bar{\phi}$	$\bar{\eta}$	τ	Ω	E	L	λ	K
M	1	0	1	0	0	0	2	2	0	0	3	3	4	4	2	2
$\mathcal{M}$	0	1	1	2	-1	0	-1	0	-2	-1	-2	-1	-3	-2	-1	0

3.5.1   Ward identities

As already mentioned a couple of times by now, TYM theories are very symmetric. This translates to the quantum theory to a very rich set of Ward identities that we will now display.

We first start with the traditional gauge fixing equation

$$\frac{\delta\Gamma}{\delta b} = d \star A \,, \tag{3.24}$$

followed by the FP antighost equation

$$\mathcal{G}_{\bar{c}}\left(\Gamma\right) = d \star \psi \,, \tag{3.25}$$

the topological ghost gauge fixing equation,

$$\frac{\delta\Gamma}{\delta\bar{\eta}} = d \star \psi \,, \tag{3.26}$$

the bosonic antighost equation,

$$\mathcal{G}_{\bar{\phi}}\left(\Gamma\right) = 0 \,, \tag{3.27}$$

where

$$\mathcal{G}_{\bar{\phi}} \equiv \left(\frac{\delta}{\delta\bar{\phi}} - d \star \frac{\delta}{\delta\tau}\right) \,, \tag{3.28}$$

and Slavnov-Taylor identity

$$\mathcal{S}\left(\Gamma\right) = 0 \,, \tag{3.29}$$

where the Slavnov-Taylor operator is now given by

$$\mathcal{S} \equiv \int \mathrm{Tr}\left[\left(\psi - \frac{\delta}{\delta\Omega}\right)\frac{\delta}{\delta A} + \left(\phi - \frac{\delta}{\delta L}\right)\frac{\delta}{\delta c} - \frac{\delta}{\delta\tau}\frac{\delta}{\delta\psi} - \frac{\delta}{\delta E}\frac{\delta}{\delta\phi} + b\frac{\delta}{\delta\bar{c}} + \mathrm{B}\frac{\delta}{\delta\bar{\chi}} + \right.$$
$$\left. + \bar{\eta}\frac{\delta}{\delta\bar{\phi}} + \Omega\frac{\delta}{\delta\tau} + \mathrm{L}\frac{\delta}{\delta E} + \mathrm{K}\frac{\delta}{\delta\lambda}\right] \,, \tag{3.30}$$

and its linearized version by

$$\mathcal{S}_{\Gamma} \equiv \int \mathrm{Tr}\left[\left(\psi - \frac{\delta\Gamma}{\delta\Omega}\right)\frac{\delta}{\delta A} - \frac{\delta\Gamma}{\delta A}\frac{\delta}{\delta\Omega} + \left(\phi - \frac{\delta\Gamma}{\delta L}\right)\frac{\delta}{\delta c} - \frac{\delta\Gamma}{\delta c}\frac{\delta}{\delta L} - \frac{\delta\Gamma}{\delta\tau}\frac{\delta}{\delta\psi} - \frac{\delta\Gamma}{\delta\psi}\frac{\delta}{\delta\tau} + \right.$$
$$\left. - \frac{\delta\Gamma}{\delta E}\frac{\delta}{\delta\phi} - \frac{\delta\Gamma}{\delta\phi}\frac{\delta}{\delta E} + b\frac{\delta}{\delta\bar{c}} + \mathrm{B}\frac{\delta}{\delta\bar{\chi}} + \bar{\eta}\frac{\delta}{\delta\bar{\phi}} + \Omega\frac{\delta}{\delta\tau} + \mathrm{L}\frac{\delta}{\delta E} + \mathrm{K}\frac{\delta}{\delta\lambda}\right] \,. \tag{3.31}$$

Finally, we also have the topological ghost equation

$$\mathcal{G}_\phi\left(\Gamma\right) = \Delta_\phi \,, \tag{3.32}$$

where

$$\mathcal{G}_\phi \equiv \int \left(\frac{\delta}{\delta\phi} - \left[\bar{\phi}, \frac{\delta}{\delta b}\right] \right) \tag{3.33}$$

and

$$\Delta_\phi \equiv \int \left([A, \tau] + [c, E] + [\bar{\chi}, \lambda]\right) \,, \tag{3.34}$$

and the first FP ghost equation

$$\mathcal{G}_c^1\left(\Gamma\right) = \Delta_c \,, \tag{3.35}$$

where

$$\mathcal{G}_c^1 \equiv \int \left(\frac{\delta}{\delta c} - \left[\bar{c}, \frac{\delta}{\delta b}\right] - \left[\bar{\chi}, \frac{\delta}{\delta B}\right] - \left[\bar{\phi}, \frac{\delta}{\delta\bar{\eta}}\right] - \left[\lambda, \frac{\delta}{\delta K}\right] \right) \,, \tag{3.36}$$

and

$$\Delta_c \equiv \int \left([A, \Omega] + [\tau, \psi] + [c, L] + [E, \phi] + [\bar{\chi}, K] + [B, \lambda]\right) \,, \tag{3.37}$$

and the second one

$$\mathcal{G}_c^2\left(\Gamma\right) = \Delta_c \,, \tag{3.38}$$

where

$$\mathcal{G}_c^2 \equiv \int \left(\frac{\delta}{\delta c} - \left[\bar{\phi}, \frac{\delta}{\delta\bar{c}}\right] + \left[A, \frac{\delta}{\delta\psi}\right] + \left[c, \frac{\delta}{\delta\phi}\right] + \left[\bar{\eta}, \frac{\delta}{\delta b}\right] + \left[E, \frac{\delta}{\delta L}\right] + \left[\tau, \frac{\delta}{\delta\Omega}\right] \right) \,, \tag{3.39}$$

and has exactly the same linear breaking Δ_c of $\mathcal{G}_c^1$. We continue with the vectorial supersymmetry

$$\mathcal{W}\left(\Gamma\right) = 0 \,, \tag{3.40}$$

where

$$\mathcal{W} \equiv \int \mathrm{Tr}\left[dA\frac{\delta}{\delta\psi} + dc\frac{\delta}{\delta\phi} + d\bar{\chi}\frac{\delta}{\delta B} + d\bar{\phi}\left(\frac{\delta}{\delta\bar{\eta}} + \frac{\delta}{\delta\bar{c}}\right) + d\left(\bar{c} + \bar{\eta}\right)\frac{\delta}{\delta b} + d\tau\frac{\delta}{\delta\Omega} + \right.$$
$$\left. + dE\frac{\delta}{\delta L} + d\lambda\frac{\delta}{\delta K} \right] \,, \tag{3.41}$$

followed by the bosonic nonlinear supersymmetry

$$\mathcal{T}\left(\Gamma\right) = 0 \,, \tag{3.42}$$

where

$$\mathcal{T} \equiv \int \mathrm{Tr} \left[\frac{\delta}{\delta\Omega} \frac{\delta}{\delta\psi} + \frac{\delta}{\delta L} \frac{\delta}{\delta\phi} + \frac{\delta}{\delta K} \frac{\delta}{\delta B} + (\bar{c} + \bar{\eta}) \left(\frac{\delta}{\delta\bar{c}} + \frac{\delta}{\delta\bar{\eta}} \right) \right] , \tag{3.43}$$

and its linearized version is given by

$$\mathcal{T}_{\Gamma} \equiv \int \mathrm{Tr} \left[\frac{\delta\Gamma}{\delta\Omega} \frac{\delta}{\delta\psi} + \frac{\delta\Gamma}{\delta\psi} \frac{\delta}{\delta\Omega} + \frac{\delta\Gamma}{\delta L} \frac{\delta}{\delta\phi} + \frac{\delta\Gamma}{\delta\phi} \frac{\delta}{\delta L} + \frac{\delta\Gamma}{\delta K} \frac{\delta}{\delta B} + \frac{\delta\Gamma}{\delta B} \frac{\delta}{\delta K} + \right.$$
$$\left. + (\bar{c} + \bar{\eta}) \left(\frac{\delta}{\delta\bar{c}} + \frac{\delta}{\delta\bar{\eta}} \right) \right] . \tag{3.44}$$

Finally, we complete the set of Ward identities of TYM with the ghost supersymmetry

$$\mathcal{G}_s (\Gamma) = 0 , \tag{3.45}$$

where

$$\mathcal{G}_s \equiv \int \left[\bar{\phi} \left(\frac{\delta}{\delta\bar{\eta}} + \frac{\delta}{\delta\bar{c}} \right) + c \frac{\delta}{\delta\phi} + \tau \frac{\delta}{\delta\Omega} + 2E \frac{\delta}{\delta L} + \lambda \frac{\delta}{\delta K} \right] , \tag{3.46}$$

It is important to state the Ward identities (3.42) and (3.45) are novel results obtained by the author and collaborators in [109] and which result in a simplification of the renormalizability features of TYM theory in the (A)SDL gauge. More specifically, the bosonic nonlinear supersymmetry kills three counterterms present in previous works [110, 21, 22]. As a consequence, the number of independent renormalizations drops from 4 to just 1.

3.5.2   Counterterms

Considering the quantum action at 1-loop

$$\Gamma^{(1)} = \Sigma + \epsilon \Sigma^{\mathrm{ct}} , \tag{3.47}$$

the set of Ward identities resume to

$$\frac{\delta \Sigma^{\text{ct}}}{\delta b} = 0 \, , \tag{3.48a}$$

$$\mathcal{G}_{\bar{c}}\left(\Sigma^{\text{ct}}\right) = 0 \, , \tag{3.48b}$$

$$\frac{\delta \Sigma^{\text{ct}}}{\delta \bar{\eta}} = 0 \, , \tag{3.48c}$$

$$\mathcal{G}_{\bar{\phi}}\left(\Sigma^{\text{ct}}\right) = 0 \, , \tag{3.48d}$$

$$\mathcal{S}_{\Sigma}\left(\Sigma^{\text{ct}}\right) = 0 \, , \tag{3.48e}$$

$$\mathcal{G}_{\phi}\left(\Sigma^{\text{ct}}\right) = 0 \, , \tag{3.48f}$$

$$\mathcal{G}_{c}^{1}\left(\Sigma^{\text{ct}}\right) = 0 \, , \tag{3.48g}$$

$$\mathcal{G}_{c}^{2}\left(\Sigma^{\text{ct}}\right) = 0 \, , \tag{3.48h}$$

$$\mathcal{W}\left(\Sigma^{\text{ct}}\right) = 0 \, , \tag{3.48i}$$

$$\mathcal{T}_{\bar{\Sigma}}\left(\Sigma^{\text{ct}}\right) = 0 \, , \tag{3.48j}$$

$$\mathcal{G}_{s}\left(\Sigma^{\text{ct}}\right) = 0 \, . \tag{3.48k}$$

Its solution

$$\Sigma^{\text{ct}} = a \int \text{Tr} \left(\text{B} \star \text{F} + 2\bar{\chi} \star \text{D}\psi + \bar{\chi} \star [c, \text{F}] \right) \tag{3.49}$$

is the most general counterterm allowed. Again, this is a novel result due to the discovery of the nonlinear bosonic supersymmetric $\mathcal{T}$-symmetry given by (3.42)

3.5.3 Quantum stability

Finally, one can show that Σ^{ct} can be absorbed in Σ by a redefinition

$$\Phi_0 \equiv z_{\Phi} \Phi \, , \tag{3.50a}$$

$$g_0 \equiv z_g g \, , \tag{3.50b}$$

$$\mathcal{J}_0 \equiv z_{\mathcal{J}} \mathcal{J} \, , \tag{3.50c}$$

of fields $\Phi \in \left\{ \text{A}, c, \psi, \phi, \bar{c}, b, \bar{\chi}, \text{B}, \bar{\phi}, \bar{\eta} \right\}$, parameter g and external sources $\mathcal{J} \in \{\tau, \Omega, \text{E}, \text{L}, \lambda, \text{K}\}$. In other words, that

$$\Sigma\left[\Phi, g, \mathcal{J}\right] + \epsilon \Sigma^{ct}\left[\Phi, g, \mathcal{J}\right] = \Sigma\left[\Phi_0, g_0, \mathcal{J}_0\right] \, . \tag{3.51}$$

In particular, the nontrivial z-factors can be evaluated as

$$z_{\text{B}} z_{\text{A}} = z_c z_{\bar{\chi}} = 1 + \epsilon a \,, \tag{3.52a}$$

$$z_{\text{E}} = z_g^{-2} z_c^{-3} \,, \tag{3.52b}$$

$$z_{\text{A}} = z_b = z_g^{-1} \,, \tag{3.52c}$$

$$z_{\text{K}} = z_g^{-1} z_c^{-1} z_{\bar{\chi}}^{-1} \,, \tag{3.52d}$$

$$z_{\Lambda} = z_g^{-2} z_c^{-2} z_{\bar{\chi}}^{-1} \,, \tag{3.52e}$$

$$z_{\bar{c}} = z_{\Omega} = z_{\psi} = z_{\bar{\eta}} = z_c^{-1} \,, \tag{3.52f}$$

$$z_{\phi} = z_{\bar{\phi}} = z_{\text{L}} = z_{\tau} = z_g^{-1} z_c^{-2} \,, \tag{3.52g}$$

clearly, the theory has only one independent renormalization. Nontheless, this constitute a system of fifteen equations and seventen unknowns, making it impossible to unambiguosly determine each z-factor.

3.6 Absense of radiative corrections

We end up this chapter by stating another novel result obtained by the author and collaborators in [111]. It was shown in [109], the propagator of the gauge field vanishes exactly

$$\langle \text{A}(x)\text{A}(y) \rangle = 0 \,, \tag{3.53}$$

to all orders in the loop expansion. This is a direct consequence of the vectorial supersymmetry $\mathcal{W}$, which is very characteristic of TQFTs in general [112].

Let us now consider the Feynman rules of TYM in the (A)SDL gauge,

$$\langle \text{AA} \rangle = \text{\scriptsize eeeeee} \quad , \qquad \langle c\bar{c} \rangle = \text{\scriptsize..........} \quad , \qquad \langle \bar{\chi}\psi \rangle = \text{\scriptsize www} \quad , \qquad \langle \text{A}b \rangle = \text{\scriptsize eee---} \quad ,$$

$$\langle \bar{\eta}\psi \rangle = \text{\scriptsize ---www} \quad , \qquad \langle \text{AB} \rangle = \text{\scriptsize eee===} \quad , \qquad \langle \phi\bar{\phi} \rangle = \text{\scriptsize ::::::::::} \quad . \tag{3.54}$$

where the relevant vertexes are represented by:

$$\tag{3.55}$$

To show the absence of radiative corrections, it is convenient to split the argumentation into propositions.

Proposition 1 *Any connected loop diagram containing an internal* A*-leg vanishes unless the branch generated by the* A*-leg ends up in external* B*- or* b*-legs.*

To prove this proposition, we must consider a combination of two facts: 1) $\langle AA \rangle = 0$ to all orders and 2) the gauge field only propagates through the non-vanishing mixed propagators $\langle BA \rangle$ and $\langle bA \rangle$. Hence, from an internal A-leg arising from an arbitrary vertex, denoted by a black dot, , we only have two possibilities: and . In the same way, the fields B and b only propagate through A. Graphically, we now have and . Nonetheless, the former is not at our disposal since there is no vertex containing b, *vide* (3.55). The latter, on the other hand, must be a BAA vertex since it is the only one containing B. Thus, an internal A-leg in any loop diagram will propagate to B and the latter will end up in a BAA vertex,

$$\tag{3.56}$$

Applying the above reasoning for the two newly created A-legs, we end up with two more BAA vertices and four A-legs. Since the number of A-legs only increases, we can continue

this process *ad infinitum* leading to a cascade effect of exponential proliferation of A-legs:


(3.57)
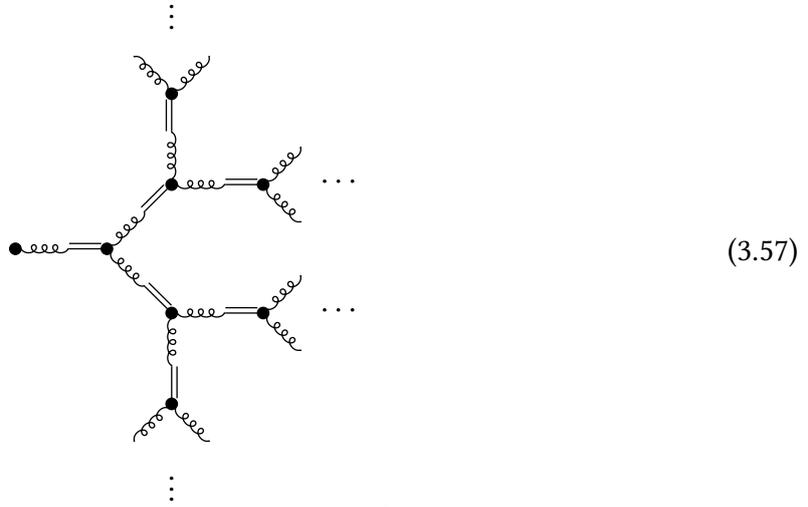
There are three possibilities here: 1) trying to close a loop in the diagram (3.57) requires an $\langle AA \rangle$ internal propagator, which would result in a vanishing diagram; 2) to consider external A-legs, which also requires a $\langle AA \rangle$ propagator, resulting in a vanishing diagram and; 3) one could consider that all remaining A-legs end up in external B- or *b*-legs.

We should note that all vertexes, except one, contain at least one A-leg, therefore the cascade effect always occur for these cases. The only exception is the vertex $\bar{\phi}c\psi$.

Corollary 1.1 *In a connected loop diagram, any branch arising from the vertex $\bar{\phi}c\psi$ results in a vanishing diagram unless this branch ends up in external* B- *or* b-*legs.*

Let us start with the vertex of interest, *i.e.* $\bar{\phi}c\psi$. To construct a loop diagram from this three-vertex we have to propagate it to another vertex. The $\bar{\phi}$-leg could only propagate to the vertex $\bar{\phi}A\phi$ through $\langle \bar{\phi}\phi \rangle$; the *c*-leg only to $\bar{c}Ac$ through $\langle \bar{c}c \rangle$ and; the ψ-leg to the vertexes $\bar{\chi}A\psi$, $\bar{\chi}cA$ or $\bar{\chi}cAA$ through $\langle \psi\bar{\chi} \rangle$ ($\langle \bar{\eta}\psi \rangle$ is not considered because there is no vertex containing $\bar{\eta}$). Graphically, the possibilities of completing the legs arising from this vertex are

$,$ $,$ $.$ (3.58)
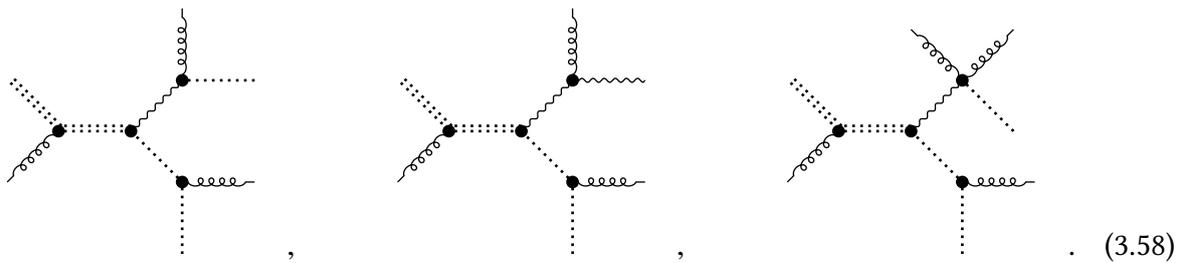
But all possible branches contain at least one remaining A-leg. By evoking Proposition 1, the proof is completed.

Corollary 1.2 *Any connected loop diagram containing a $(\Phi_i \neq \{B, b\})$-external leg vanishes.*

There are two steps toward this proof: 1) consider the external leg joined to a vertex containing an A field. In this case, A is an internal leg. Thus, Proposition 1 takes place and the graph either vanishes or generates a branch with external B- or b-legs and no loop can be constructed; 2) now, consider the external leg joined to a vertex not containing A, *i.e.* the vertex $\bar{\phi}c\psi$. The field $\bar{\phi}$ only propagates through $\langle\bar{\phi}\phi\rangle$, c through $\langle\bar{c}c\rangle$, and ψ only through $\langle\bar{\chi}\psi\rangle$ or $\langle\bar{\eta}\psi\rangle$. For this reason, it is impossible to propagate the vertex $\bar{\phi}c\psi$ to another vertex $\bar{\phi}c\psi$. In other words, from the vertex $\bar{\phi}c\psi$, we should necessarily propagate it to the vertexes containing an A field. Now, Corollary 1.1 takes place and the graph, again, either vanishes or generates a branch with external B- or b-legs and no loop can be constructed.

Proposition 2 *Any connected n-point function of the form $\langle B(x_1)B(x_2)...b(x_{n-1})b(x_n)\rangle$ vanishes.*

Due to the doublet strucute of B and b, and the fact that expectation values of any BRST-exact terms vanish. One can write these n-functions as BRST-exact correlators, namely,

$$\langle BBB\ldots bb\rangle = \langle s\bar{\chi}BB\ldots bb\rangle = \langle s(\bar{\chi}BB\ldots bb)\rangle = 0\,, \tag{3.59}$$

and

$$\langle BBB\ldots bb\rangle = \langle BB\ldots s\bar{c}b\rangle = \langle s(BBB\ldots \bar{c}bb)\rangle = 0\,, \tag{3.60}$$

which vanish due to BRST-invariance.

Proposition 3 *All connected n-point Green functions are tree-level exact.*

Let us take a connected loop diagram with n external legs with arbitrary fields Φ_i. From Corollary 1.2, if there is at least one field different from B or b, the graph either vanishes or is a tree-level graph. Then, there remains the possibility of a graph with n external legs formed by B or b fields. In this case Proposition 2 takes over and the Green function $\langle BB\ldots bb\rangle$ vanishes, meaning that this Green function is zero and receive no radiative corrections. Hence, all connected n-point Green functions are tree-level exact.

This result should not strike the reader as a surprise. Remember that the path integral of TYM theory in independent of the coupling parameter g. Thus, it is to be expected that the tree-level approximation is actually exact.

Chapter 4

Gravity

4.1 Einstein's General Theory of Relativity

4.1.1 Introduction

The year 1905 was A. Einstein's *annus mirabilis*. This is a latin phrase that translates to "miraculous year"[1]. In this year, Einstein published a series of four geniuses papers that revolutionized our understanding of Nature.

The first one, about the photoelectric effect [113], was pivotal to the early develpment of quantum theory. The second one, about the Brownian motion [114], gave credible evidence on the existence of molecules and the discreteness of matter. On the third, he proposed a reconciliation between the laws of motion and the Maxwellian electrodynamics [115] and, finally, on the fourth, he derived the mass-energy equivalence [116] relation[2]. In particular, these last two papers represent the birth of his Special Theory of Relativity (STR) and of relativistic field theories in general.

The STR has origin in Einstein's realization that Maxwell's equations hold valid in an equivalence class of frames of references that more accurately captured the isometries of space and time in the absence of gravity. Differently from Newtonian physics, frames in this equivalence class were connected by representations of the Lorentz group $SO(1, 3)$. Consequently, relativistic laws were more adequately written in an explicitly $SO(1, 3)$ covariant fashion.

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[1]This expression was originally used to refer to 1666. This was the year Sir Isaac Newton developed the fundations of his corpuscular theory of light, calculus, his laws of mechanics and gravitation, while isolated for two year in his country home, near Lincolnshire, due to the Great Plague that devastated England during 1665 and 1666.

[2]These four articles can be found translated to english in reference [117].

In the following years, the search for the deeper meaning behind Einstein's kinematic statements caught a lot of attention. By the works of H. Minkowski, V. Ignatowsky, P. Frank, H. Rothe, A. D. Aleksandrov and others[3], it became clear that the STR could be understood in a purely geometrical way. The STR is nothing but the adoption of a background arena that fundamentally differs from the tradicional Galilean space(time). In particular, its geometry determines a causal structure that allows only for local interactions[4]. Nonlocal interactions, like the action at a distance from Newtonian mechanics, were approximations valid in a regime in which light signals travel instantaneously, i.e., the speed of light is infinite.

The incompatibility between STR and Newton's law of universal gravitation was very clear from the begining. Einstein, in particular, immediately started working out the problem. In 1907, his great physical intuition allowed him to realize that the Galilean equivalence principle[5] implied that, locally, a body could not physically distinguish gravity from inertial forces and vice-versa. He then drew the conclusion that any frame of reference is as good as Lorentzian ones in the local description of relativistic physics.

Einstein developed his relativistic theory of gravity with such principle in mind, making it manifestly covariant under general coordinate transformations. This is why he gave it the name of General Theory of Relativity or General Relativity (GR), for short. After eight year of trial and error, he finally concluded his quest and published his results in the November 1915 paper [119].

4.1.2 Background independence

The deeper meaning behind GR, however, was still an open debate. In 1917, it was very clear for physicist E. Kretschmann [120] that Einstein's "principle of general covariance" could not be used as the defining feature of GR, nor any physical theory for that matter.

The reason behind Kretschmann's criticism is that coordinate systems are just labels that we give to small regions of spacetime. Coordinate transformations are just the relabelling of these regions. This alone cannot carry any physicality. Every respectable physical theory must be independent of how we decide to label spacetime events. Therefore, all physical theories, relativistic or not, must be able to be written in a generally covariant form.

E. Cartan did exactly that to Newtonian gravity in his 1922 and 1923 papers [121, 122], for example. In its generally covariant form, Newtonian gravity can also be understood as the

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[3]A brief historical account on the axiomatic formulation of STR can be found in [118], chapter 2.

[4]An interaction is said to be local when it immediately affects only its neighbourhood.

[5]Also known as the universal principle of free fall, it states the equality between inertial and gravitational mass. Two bodies, or arbitrary masses, subjected to the same gravitational field will experience the same acceleration.

curvature of an affine connection: it is just a different (non-Riemannian) one[6]. Again, the real difference between nonrelativistic and relativistic theories does not lie on the geometrical framework, but on their causal structure: the latter has light cones, event horizons, etc, while the former does not.

Many authors[7] argue today that Einstein's ideas about spacetime, gravity and general covariance, should be more modernly understood as an equivalence class of 4-manifolds M, related by the action of the nonabelian group of diffeomorphisms $Diff$ (M). In other words, that the symmetry principle of diffeomorphism invariance[8] is the key ingredient of GR. This symmetry is undoubtly present in GR. Nonetheless, as it is very well argued in [123], we can also write any physical theory in a diffeomorphism invariant fashion, making this another physically trivial symmetry.

If not general covariance nor diffeomorphism invariance, what is then so special about GR? Well, differently from its predecessors, GR actually lacks a formulation that is **not** generally covariant and diffeomorphism invariant. In fact, Einstein's realization that a family of bilinear forms (a metric) exists in spacetime that determines all of its geometric properties and, in particular, that gravity is its curvature[9]. Had, as an immediate consequence, that (almost) no aspects of the (Riemannian) geometry of spacetime is left fixed or immutable: they are all dynamically determined by an action principle. This (partial) lack of an *à priori* geometry is now known as background independence.

In despite of Einstein's intent and modern misconceptions, GR is not the pinacle of a symmetry principle (these are gauge theories!). It is *de facto* the first prototype of a background indepedent theory. General covariance and diffeomorphism invariance are just inevitable consequences of this fact, not the other way around. Inadvertently, Einstein painted all these concepts with a thick brush of "general covariance" and, as C. Misner, K. Thorne and J. Wheeler put it, "fathered ~~half~~ a century of confusion" [125].

To be clear, full background independence means a total lack of "a background", i.e., fixed features of spacetime. This includes dimension, topology, smooth structure and geometry of the manifold M: all must be dynamically determined. In this sense GR, as well as TQFT, are only partially background independent theories: GR for it only determines geometry; TQFT for it only determines smooth structure or topology.

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[6]See [?] for a detailed review.

[7]Mainly C. Rovelli and L. Smolin. But it seems to be a widespread believe in the community, see [123] and references therein.

[8]Here we adopt the precise definition of diffeomorphism invariance as given by [124]. In particular, the reader should take special care to not mistakenly exchange it for diffeomorphism covariance, also defined in [124], which is just a fancy name for the old general covariance.

[9]GR is now classified as a "metric theory of gravity" because of that fact.

4.1.3   Riemannian geometry

If, on one hand, Einstein's understanding of the metric tensor $g_{\mu\nu}$ as the only dynamical field responsable for gravitational effects paved the road to background independence. On the other, this very assumption carried consequences that unecessarily crippled this very concept.

For $g_{\mu\nu}$ to determine the affine structure of spacetime, encoded in the affine connection $\Gamma^{\alpha}{}_{\mu\nu}$, one has to assume that i) the nonmetricity tensor $Q_{\alpha\mu\nu}$ is zero

$$Q_{\alpha\mu\nu} \equiv \nabla_{\alpha} g_{\mu\nu} = 0 \,, \tag{4.1}$$

and ii) the torsion tensor $T^{\alpha}{}_{\mu\nu}$ is zero

$$T^{\alpha}{}_{\mu\nu} \equiv \Gamma^{\alpha}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\nu\mu} = 0 \,. \tag{4.2}$$

These equations do not come from an action principle. In fact, they have to be assumed or imposed. Therefore, they constitute a fixed part of the geometry, i.e., "a background". In the following sections, we will discuss theories that naturally generalize GR by removing such background. In this sense, they represent a better implementation of the background independence principle.

In the presence of a (pseudo-)Riemannian metric $g_{\mu\nu}$, the affine connection admits a general splitting

$$\Gamma^{\alpha}{}_{\mu\nu} = \mathring{\Gamma}^{\alpha}{}_{\mu\nu} + K^{\alpha}{}_{\mu\nu} \,, \tag{4.3}$$

where

$$\mathring{\Gamma}^{\alpha}{}_{\mu\nu} \equiv \frac{1}{2} g^{\alpha\beta} \left(-\partial_{\beta} g_{\mu\nu} + \partial_{\mu} g_{\nu\beta} + \partial_{\nu} g_{\beta\mu} \right) \,, \tag{4.4}$$

is its Riemannian part[10], known as Levi-Civita connection or Christoffel symbols, and

$$K^{\alpha}{}_{\mu\nu} \equiv \frac{1}{2} Q^{\alpha}{}_{\mu\nu} - Q_{(\mu\nu)}{}^{\alpha} + 2T_{(\mu\nu)}{}^{\alpha} + T^{\alpha}{}_{\mu\nu} \,, \tag{4.5}$$

is its non-Riemannian part, known as the distortion tensor[11]. Again, the presence of the background (4.1) and (4.2) vanishes the distortion tensor $K^{\alpha}{}_{\mu\nu}$, making the affine connection $\Gamma^{\alpha}{}_{\mu\nu}$ purely Riemannian, as envisioned Einstein.

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[10]From now on, we will use the overhead symbol ˚ to denote purely Riemannian quantities.

[11]Here we should clarify the notation:

$$Q_{(\mu\nu)}{}^{\alpha} \equiv \frac{1}{2!} \left(Q_{\mu\nu}{}^{\alpha} + Q_{\nu\mu}{}^{\alpha} \right) \tag{4.6}$$

and so on.

4.1.4 Dynamics

To determine the dynamics of the Riemannian geometries, we evoke a simple but very powerful theorem by D. Lovelock [126]: under the assumption of iii) locality, iv) polynomiality and; v) that the resulting Euler-Langrange equations consists of a Cauchy problem, it states the most general action functional for $g_{\mu\nu}$, in d dimensions[12]. For $d = 4$, in particular, we have

$$S\left[g_{\mu\nu}\right] = \int d^4x\sqrt{-g}\left[\alpha_0 + \alpha_1\mathring{R} + \alpha_2\left(\mathring{R}^2 - 4\mathring{R}_{\mu\nu}\mathring{R}^{\mu\nu} + \mathring{R}_{\alpha\beta\mu\nu}\mathring{R}^{\alpha\beta\mu\nu}\right) + \right.$$
$$\left. + \alpha_3\epsilon_{\mu\nu\lambda\sigma}\mathring{R}^{\alpha\beta\mu\nu}\mathring{R}_{\alpha\beta}{}^{\lambda\sigma}\right], \tag{4.7}$$

where the α's are arbitrary coupling parameters of mass dimension 4, 2 and 0, respectively. It should be clear that $\mathring{R}^\alpha{}_{\beta\mu\nu}$ are the components of the Riemannian curvature tensor, $\mathring{R}_{\mu\nu}$ its partial trace and $\mathring{R}$ its full trace.

The coupling α_2 and α_3 in (4.7) are the spacetime representation of the Gauss-Bonnet and Pontryagin densities, respectively. As we already discussed, these terms are characteristic classes that classify the smooth structure of 4-manifolds. Consequently, their integrals are global invariant that do not contribute to the local dynamics enconded in the Euler-Lagrange equations[13].

The α_1 coupling in (4.7) is the well-known Einstein-Hilbert (EH) term, first derived by D. Hilbert in November 1915 [127], and α_0 is the cosmological constant term. In summary, the EH and cosmological constant term

$$S\left[g_{\mu\nu}\right] = \int d^4x\sqrt{-g}\left(\alpha_0 + \alpha_1\mathring{R}\right), \tag{4.8}$$

encode all the local dynamics of GR.

The universality of gravity contained within the Galilean equivalence principle is, of course, inherited by GR. In fact, matter fields - here collectively denoted by Φ_μ - couple minimally to $\mathring{\Gamma}^\alpha_{\mu\nu}$ as a consequence of the mathematical formalism alone; no additional physical principle is needed. If the Lagrangian $\mathcal{L}(\Phi_a, \partial_b\Phi_a)$ describes the dynamics of the field Φ in an inertial coordinate system x^a, then general covariance demands this same dynamics to be described by Lagrangian $\mathcal{L}(\Phi_\mu, \mathring{\nabla}_\nu\Phi_\mu)$ in a noninertial coordinate system x^μ. The dynamics

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[12]This theorem defines the the Lovelock family of gravity theories. It consists of the most natural generalizations of GR for higher spacetime dimensions. In particular, it coincides with GR dynamics for dimensions lower than five.

[13]This result is very peculiar to 4-manifolds. In general, the integral of the Gauss-Bonnet density is not a topological invariant. Indeed, for higher dimensional manifolds it represents an important ultraviolet correction to Einstein-Hilbert dynamics.

of the gravitational and matter fields coupled together is given by

$$S\left[g_{\mu\nu}, \Phi_\mu\right] = S\left[g_{\mu\nu}\right] + \int d^4x\sqrt{-g}\mathcal{L}\left(\Phi_\mu, \mathring{\nabla}_\mu\Phi_\nu\right) . \tag{4.9}$$

The Euler-Lagrange equations for the gravitational field described by action (4.9) are

$$-\frac{1}{2}\alpha_0 g_{\mu\nu} + \alpha_1\left(\mathring{R}_{\mu\nu} - \frac{1}{2}\mathring{R}g_{\mu\nu}\right) - \frac{1}{2}\tau_{\mu\nu} = 0 , \tag{4.10}$$

where $\tau_{\mu\nu}$ is the total energy-momentum tensor of matter field Φ_μ. We can cast them in their most standard form by evoking the Correspondence Principle[14]. This will fix α_1 in term of Newton's constant G; indeed, $\alpha_1 = (16\pi G)^{-1}$. We can also define $-\alpha_0/2\alpha_1$ as Λ^2, which we call the cosmological constant, and $\mathring{R}_{\mu\nu} - (1/2)\mathring{R}g_{\mu\nu}$ as $\mathring{G}_{\mu\nu}$, which we call the Einstein's tensor. The result is

$$\mathring{G}_{\mu\nu} + \Lambda^2 g_{\mu\nu} = 8\pi G\tau_{\mu\nu} , \tag{4.11}$$

also known as Einstein's field equations[15].

On the other hand, the Euler-Lagrange equations for the matter field Φ are

$$\frac{\partial\mathcal{L}}{\partial\Phi_\mu} - \mathring{\nabla}_\nu\left[\frac{\partial\mathcal{L}}{\partial\left(\mathring{\nabla}_\nu\Phi_\mu\right)}\right] = 0 , \tag{4.12}$$

which is nothing but the traditional Euler-Lagrange equations written in a generally covariant form.

Field equations (4.11) and (4.12) are coupled together and cannot be solved separately because of their nonlinearities. Indeed, they are extremely difficult to solve. Even if Φ_μ is not considered part of the system, so we can disregard (4.12) and interpret $\tau_{\mu\nu}$ as an external source, a general analytical solution still lacks.

The main difficult in solving Einstein' field equations is exactly because GR lacks a noncovariant formulation. Differently from Newtonian gravity's covariant field equations,

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[14]It states that the Newtonian gravity should be recovered from GR in some appropriated limit. The correct limiting procedure, however, is much more tricky than the reader might imagine. It was first clarified by the works of E. Cartan [122, 128] and K. Friedrichs [129] in the twienties. It does not envolve a weak field limit since, as already mentioned, Newton's theory itself can be casted in a generally covariant form. Again, a detailed discussion can be found in [?].

[15]They were published by Einstein in November 1915 [119], but also by Hilbert [127]. Historically, it is not clear who derived them first. However, a few year later, Hilbert gave all the credits to Einstein and any major controversy about this issue ceased.

([4.11](#)) must be solved as it stands: a nonlinear coupled set of 10 partial differential equations of second order.

Particular solutions, on the other hand, can be obtained fairly easy - specially for highly symmetric cases - such as the Schwarzschild metric, the Friedmann-Lemaître-Robertson-Walker metric, and so on. In fact, one can arguably say that every Lorentzian manifold[16] is a solution to Einstein's field equation with an appropriated $\tau_{\mu\nu}$ as external source[17].

4.2 The Einstein-Cartan theory

4.2.1 Introduction

Geometric concepts such as length, angle, area, volume, etc only exist in spaces with a well defined metric. On the other hand, geometric concepts such as parallelism need an affine connection.

Mathematician E. Cartan was the first to segregate these concepts into two logically independent classes, namely, metricity and affinity [121]. Of course, we are not forbidden to reduce parallelism to just a mere measurement of angles. In fact, this is the essence of the Riemannian geometry: all geometric features reduced to metricity. It, nonetheless, represents a very particular scenario.

The distinction between metricity and affinity started a debate between Einstein and Cartan. On one side, Einstein advocated for the economy of fundamental fields and thus for a Riemannian geometry for spacetime. On the other, Cartan argued for the logical independence of the metric tensor and affine connection [121].

In Cartan's view, it is not necessary to assume *à priori* a vanishing torsion and/or non-metricity tensor. Though less economic in fundamental fields, Cartan's approach is more economic in assumptions about geometrical features of spacetime. Indeed, it eliminates the background (4.1) and (4.2) of GR and, as such, represents an improved implementation of the background independence principle.

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[16]A Lorentzian manifold is a special kind of (pseudo-)Riemannian manifold in which the signature of the metric is $(1, n-1)$ in n dimensions. Indeed, this is the only kind of (pseudo-)Riemannian manifold that we are considering here.

[17]The reasoning is the following: given a Lorentzian metric, evaluate its Einstein's tensor - which is purely a mathematical operation. Divide the result by $8\pi G$ and declare it the external source $\tau_{\mu\nu}$. The "arguably" comes from the debate about this scenario being physically relevant or not. More precisely, if the resulting energy-momentum $\tau_{\mu\nu}$ describes a physically reasonable matter content.

4.2.2   Non-Riemannian geometry

Let us consider then the case of $g_{\mu\nu}$ and $\Gamma^{\alpha}_{\mu\nu}$ as independent fundamental fields. This opens up the possibility of removing background (4.1) and/or (4.2) which, in general, means a non-Riemannian geometry for spacetime.

In these scenarios, the gravitational field has more degrees of freedom than in the previous case. In particular, these extra degrees of freedom can be coupled to alternative kinds of conserved currents coming from the matter sector, e.g., spin, dilation and shear currents [130].

Perhaps the simplest examples is the Einstein-Cartan (EC) theory of gravity, in which torsion is nonvanishing and can be coupled with the spin current of matter. Let us concentrate in this theory.

In the EC theory, the background (4.2) is removed while (4.1) is kept. In other words, the affine connection $\Gamma^{\alpha}_{\mu\nu}$ has a nonvanishing $K^{\alpha}_{\mu\nu}$, which is given by purely the torsional terms,

$$\Gamma^{\alpha}_{\mu\nu} = \mathring{\Gamma}^{\alpha}_{\mu\nu} + 2T_{(\mu\nu)}{}^{\alpha} + T^{\alpha}_{\mu\nu} \ . \tag{4.13}$$

Spacetimes endowed with a metric $g_{\mu\nu}$ and the above affine connection $\Gamma^{\alpha}_{\mu\nu}$ are called Riemann-Cartan manifolds. These manifolds fundamentally differ from Riemannian ones by the fact that infinitesinal paralellograms fail to close. In particular, this failure is exactly proportional to the nonvanishing torsion tensor $T^{\alpha}_{\mu\nu}$.

We can easily verify the existence of extra degree of freedom in Riemann-Cartan manifolds. This can be done by interpreting equations (4.1) and (4.2) as actual constraints. In particular, Riemannian 4-manifolds have only 10 degrees of freedom, all coming from $g_{\mu\nu}$. This is exactly because the 64 extra degrees of freedom coming from a general $\Gamma^{\alpha}_{\mu\nu}$ were all eliminated by the backgrouns (4.1) and (4.2), each representing 40 and 24 constraints, respectively.

On the other hand, there is no background (4.2) in Riemann-Cartan manifolds, which means 24 less constraints and thus 24 surviving degrees of freedom coming from $\Gamma^{\alpha}_{\mu\nu}$. These 24 extra degrees of freedom are exactly the degrees of freedom enconded in the torsion tensor $T^{\alpha}_{\mu\nu}$, which is the antisymmetric part of $\Gamma^{\alpha}_{\mu\nu}$. In particular, these can be coupled with the antisymmetric sector of conserved currents of matter, e.g., the spin current density.

4.2.3   Dynamics

The dynamics of these degrees of freedom, in pure EC theory, is determined by a generalized version of the EH action (with possibly a cosmological constant term). Explicitly,

$$S_{EH}\left[g_{\mu\nu}, \Gamma^{\alpha}_{\mu\nu}\right] = \frac{1}{16\pi G} \int d^4x \sqrt{-g}\left(-2\Lambda^2 + R\right) \ . \tag{4.14}$$

Equation (4.14) can be seen as a generalization of (4.8) because R is now the non-Riemannian scalar curvature. It differs from $\mathring{R}$ by torsional contributions. In fact, following Cartan's interpretation, the curvature and Ricci tensors are affine concepts. In this way, they should be viewed as a function of $\Gamma^\alpha_{\mu\nu}$ and $\partial_\alpha \Gamma^\beta_{\mu\nu}$ alone; not of the metric $g_{\mu\nu}$. In other words, we must consider $R = R_{\mu\nu}(\Gamma, \partial\Gamma) g^{\mu\nu}$, which is considerably different from the Riemannian counterpart $\mathring{R} = \mathring{R}_{\mu\nu}(g, \partial g, \partial\partial g) g^{\mu\nu}$.

Again, gravity naturally couples minimally to matter as a consequence of the mathematical formalism

$$S\left[g_{\mu\nu}, \Gamma^\alpha_{\mu\nu}, \Phi_\mu\right] = S_{EH}\left[g_{\mu\nu}, \Gamma^\alpha_{\mu\nu}\right] + \int d^4x \sqrt{-g}\, \mathcal{L}\left(\Phi_\mu,\ \nabla_\mu\Phi_\nu\right) , \tag{4.15}$$

where it should be clear that ∇_μ is the covariant derivative operator with respect to the non-Riemannian connection $\Gamma^\alpha_{\mu\nu}$.

The Euler-Lagrange equations for the gravitational field, now described by g_μ and $\Gamma^\alpha_{\mu\nu}$, are, respectively,

$$G_{\mu\nu} + \Lambda^2 g_{\mu\nu} = 8\pi G \tilde{\tau}_{\mu\nu} , \tag{4.16a}$$

$$\tilde{T}^\alpha_{\mu\nu} = 8\pi G s^\alpha_{\mu\nu} , \tag{4.16b}$$

where some definitions[18] were used. In particular, $G_{\mu\nu}$ is the non-Riemannian generalization of Einstein's tensor and $\tilde{\tau}_{\mu\nu}$ the modified energy-momentum tensor. Both of these tensors are, in general, asymmetric. On the other hand, the tensor $\tilde{T}^\alpha_{\mu\nu}$, known as the modified torsion, only differs from torsion itself in its trace. Further, $s^\alpha_{\mu\nu}$ are the component of the spin density of matter.

Equations (4.16a) are clearly Einstein-like field equations in which $\tilde{\tau}_{\mu\nu}$ acts as source of curvature. Equations (4.16b), on the other hand, are known as Cartan's field equations and have the spin density $s^\alpha_{\mu\nu}$ acting as source of torsion.

Cartan's field equations (4.16b) fundamentally differ from (4.16a) by its algebraic nature. In other words, the torsion degrees of freedom do not propagate outside of the spin density: if $s^\alpha_{\mu\nu} = 0$, then $\tilde{T}^\alpha_{\mu\nu} = 0$ and, consequently, $T^\alpha_{\mu\nu} = 0$. We can also use (4.16b) to substitute $\tilde{T}^\alpha_{\mu\nu}$ for $8\pi G s^\alpha_{\mu\nu}$ everywhere. This effectively eliminates torsion from all equations, resulting

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$$\tilde{T}^\alpha_{\mu\nu} \equiv T^\alpha_{\mu\nu} + \delta^\alpha_\mu T^\beta_{\nu\beta} - \delta^\alpha_\nu T^\beta_{\mu\beta} ,$$

$$\tilde{\tau}_{\mu\nu} \equiv \tau_{\mu\nu} - \frac{1}{8\pi G} \tilde{\nabla}_\alpha \tilde{T}^\alpha_{\mu\nu} ,$$

$$\tilde{\nabla}_\mu \equiv \nabla_\mu - \tilde{T}^\nu_{\mu\nu} .$$

in

$$\mathring{G}_{\mu\nu} + \Lambda^2 g_{\mu\nu} = 8\pi G \tau_{\mu\nu} + (8\pi G)^2 \left[-4 s^{[\alpha}{}_{\beta\mu} s^{\beta]}{}_{\alpha\nu} - 2 s^{\alpha\beta}{}_{\mu} s_{\alpha\beta\nu} + s_{\mu}{}^{\alpha\beta} s_{\nu\alpha\beta} + \right.$$
$$\left. + \frac{1}{2} g_{\mu\nu} \left(4 s^{[\alpha}{}_{\beta\lambda} s^{\lambda]\beta}{}_{\alpha} + s^{\alpha\beta\lambda} s_{\alpha\beta\lambda} \right) \right] . \tag{4.17}$$

4.2.4 Advantages

Equation (4.17) is particularly useful when one wishes to compare EC theory with GR. Indeed, at the level of the field equations, the EC theory can be microscopically understood as GR corrected in its energy-momentum tensor by second order contributions of the spin density. Clearly, GR dynamics is recovered if $s^{\alpha}{}_{\mu\nu}$ vanishes.

Macroscopically, the spin density of matter tends to average to zero, $\langle s \rangle = 0$, due to the random orientation of its microscopic components. Its variance, $\Delta s = \langle s^2 \rangle$, however, may not. Then even for a macroscopically vanishing spin density tensor, we do not exactly recover GR from the averaged version of (4.17). Rather, we get $\langle \tau \rangle$ corrected by $\langle s^2 \rangle$. This correction is negligible at normal densities of matter[19]. In the early stages of our Universe, however, extremely high matter densities may have been present.

Cosmological models based on the EC theory such as [131–134] avoid the initial singularity due to a negative pressure contribution coming from $\langle s^2 \rangle$. In fact, at these extremely high densities this contribution tends to overhelm the usually attractive force of gravity. The end result is generally a bounce.

The compelling features of EC theory convinced physicists F. Hehl, P. von der Heyde and G. Kerlick to go as far as claming, in reference [135], that

> "(...) the field equations (4.17) are, at a classical level, the correct microscopic gravitational field equations. Einstein's field equations (4.11) ought to be considered a macroscopic phenomenological equation of limited validity, obtained by averaging equation (4.17)."

they continue,

> "Thus, we would propose that EC theory is a more natural starting point for a quantization program."

which is the issue we want to address.

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[19]For comparison, consider a spin fluid made out of neutrons. In this case, this correction only becomes significat at densities $\sim 10^{54}$ g cm^{-3}. The typical density of a neutron star, one the most compact objects of the Universe, is $\sim 10^{15}$ g cm^{-3}.

It is important to point out to the reader that the EC theory is the simplest modification of GR by including a nonvanishing torsion. More general theories of gravity exist for Riemann-Cartan spacetimes and beyond. For example, there exists the so-called metric-affine theories of gravity [130], in which background (4.1) is also dropped. We will, however, not present these theories on this thesis. We will restrict ourselves to the Riemann-Cartan spacetimes only.

4.3   Gauge theoretical framework

4.3.1   Introduction

The first attemps to describe gravity as a gauge theory began in the fififies and sixties. In his 1956 paper [136], R. Utiyama arrived in a gravity theory by gauging the Lorentz group $SO(1, 3)$. His approach, however, was not satisfactory.

The current conserved due to the $SO(1, 3)$ gauge symmetry is of angular momentum and/or intrinsic spin. We very well know, however, that these are not the only sources of gravity. Thus, there was an expectation that a more complete theory of gravity could be obtained by also considering spacetime translations $\mathbb{R}^4$, i.e., by gauging the Poincaré group, $ISO(1, 3) = SO(1, 3) \times \mathbb{R}^4$, instead.

The 1961 paper by T. Kibble [137] and the 1962 paper by D. W. Sciama [138] pointed exactly on that direction. In his work, Kibble directly gauged the Poincaré group while Sciama achieved similar result by following an analogy between electrical charge and spin. The resulting gravity, as expected, did have energy-momentum as one of its sources.

In gauging the Poincaré group $ISO(1, 3)$, we are forced to introduce 40 new fields in order to maintain covariance. 24 of them form an 1-form $A^{ab} = -A^{ba}$ related to the Lorentz sector, while the remaining 16 combine in the 1-form e^a related to the translational sector. We thus have

$$A^{ab} = A^{ab}{}_\mu(x)dx^\mu \,, \tag{4.18a}$$

$$e^a = e^a{}_\mu(x)dx^\mu \,, \tag{4.18b}$$

where the latin indexes also run from 0 to 3.

The disconnected nature of the Poincaré Lie group manifold[20] lead us to consider A^{ab} as the $SO(1, 3)$ piece and e^a as the $\mathbb{R}^4$ piece of the full Poincaré connection $\mathbb{A}$, i.e.,

$$\mathbb{A} = A^{ab}\sigma_{ab} + e^a P_a \,, \tag{4.19}$$

where σ_{ab} are the generators of the Lorentz algebra

$$[\sigma_{ab}, \sigma_{cd}] = \delta^{[e}{}_{(a}\delta^{f]}{}_c \, \eta_{bd)}\sigma_{ef} \,, \tag{4.20}$$

P_a are the generators of the translational algebra

$$[P_a, P_b] = 0 \,, \tag{4.21}$$

and $\eta_{ab} \equiv \mathrm{diag}\,(-1, 1, 1, 1)$ is the Minkowski metric. Together they satisfy

$$[\sigma_{ab}, P_c] = \eta_{c[a}P_{b]} \,, \tag{4.22}$$

thus closing the full Poincaré algebra.

In the search of an invariant 4-form, we associate to $\mathbb{A}$ the curvature

$$\begin{aligned}\mathbb{F} &= d\mathbb{A} + \mathbb{A}\mathbb{A} \,, \\ &= F + de + [A, e] \,, \end{aligned} \tag{4.23}$$

where $F = dA + AA$ is the curvature of A, de is the (Abelian) curvature of e and $[A, e]$ is the mixture. Besides the Pontryagin density, the only 4-form that we can built out of $\mathbb{F}$ on 4-manifolds is the YM lagrangian density,

$$\begin{aligned}\mathcal{L}_{YM} &= \mathrm{Tr}\,(\mathbb{F} \star \mathbb{F}) \,, \\ &= \mathrm{Tr}\,(F \star F + T \star T + 2F \star T) \,, \end{aligned} \tag{4.24}$$

where the definition $T \equiv de + [A, e]$ was used.

The last term in (4.24), however, breaks both Lorentz and translational symmetry. Therefore, there exists no invariant action functional for the $ISO(1, 3)$ gauge theory on 4-manifolds. In this way, the Scima-Kibble approach cannot be considered as a proper $ISO(1, 3)$ gauge theory for gravity. Again, this fact has origins on the disconnected nature of the Poincaré group, which produced the mixing $2F \star T$.

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[20]The $SO(1, 3)$ Lie group is a 6-manifold while $\mathbb{R}^4$ is a 4-manifold. Thus, $ISO(1, 3) = SO(1, 3) \times \mathbb{R}^4$ is necessarily a disconnected manifold since the dimensions of its components do not match.

It is clear from (4.24), however, that if we consider each connected sector of the Poincaré group at a time, we would have invariant 4-forms at our disposal. Thus, the Sciama-Kibble approach is able to provide two distinct gauge scenario to describe gravity: i) a $\mathbb{R}^4$ gauge theory in which e^a is a connection form while A^{ab} is just an invariant field and; ii) a SO(1, 3) gauge theory in which A^{ab} is the connection form while e^a behaves as a Lorentz vector.

The first scenario above can reproduce GR as an YM theory for translations in which the field strength is the torsion tensor and the curvature is actually zero. It is known as the teleparallel gravity [139] which, though very interesting, will be not explore in this thesis.

The second scenario is actually able to reproduce the Einstein-Cartan theory - nonvanishing curvature and torsion - in a gauge theoretical framework. This is the case we are interested in.

4.3.2 The Einstein-Cartan-Sciama-Kibble theory

The Einstein-Cartan-Sciama-Kibble (ECSK) theory is an SO(1, 3) gauge theory for gravity in which the gauge space is "soldered" to spacetime. In other words, that the internal gauge symmetries and the external spacetime symmetries are intertwined.

Consider a general coordinate transformation in a small region x of spacetime

$$x'^{\mu} = x'^{\mu}(x^{\nu}) \ . \tag{4.25}$$

A 1-forms on the cotangent bundle TM* of spacetime will feel this coordinate transformation and transform accordingly to

$$dx'^{\mu} = J^{\mu'}{}_{\nu}(x)dx^{\nu} \ , \tag{4.26}$$

where $J^{\mu'}{}_{\nu} \equiv \partial x'^{\mu}/\partial x^{\nu}$ is the Jacobian of the transformation (4.25). Clearly, $J^{\mu'}{}_{\nu}$ is a 4×4 invertible matrix, i.e., a representation of GL(4, $\mathbb{R}$) acting on the fibers of TM*. It is then natural to suppose that the gauge structure of gravity is of a GL(4, $\mathbb{R}$)-bundle in which TM* is an associated vector bundle.

The principal bundle of linear frames LM is such a GL(4, $\mathbb{R}$)-bundle $\pi : P \to M$ in which the total space P is the space of all linear frames[21] over spacetime M. A typical fiber $\pi^{-1}(x)$ over x consists of all frames θ_A that can be defined over x. A section on P is a local assignment of a frame θ_A to each point $x \in M$, i.e., a field of frames $\theta_A(x)$ over spacetime.

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[21] A linear frame θ_A is an ordered set of four linearly independent vectors that carry a linear representation of GL(4, $\mathbb{R}$) in the vector bundle V associated to LM, i.e., $\theta'_B = M_B{}^A \theta_A$; $M_B{}^A \in$ GL(4, $\mathbb{R}$) and θ_A spams V. Capital latin indexes also run from 0 to 3 but they do not feel general coordinate transformations since they are gauge (internal) indexes.

If V is a vector bundle associated to LM, then there exists a natural isomorphism $e : \text{TM} \rightarrow \text{V}$. This is the map responsible for "soldering" the gauge vector space V to the tangent bundle TM of spacetime. Thus mixing internal to external symmetries and making gravity such a peculiar gauge theory.

Using the local basis $\theta_A(x)$ of V and dx^μ of TM*, we can express e as the V-valued 1-form on M

$$e(x) = e^A{}_\mu(x) \left(\theta_A \otimes dx^\mu\right) \, , \tag{4.27}$$

more commonly known as the soldering form or the vierbein. The matrix-valued function

$$e^A{}_\mu(x) \equiv \theta^A \left(e\left(\partial_\mu\right)\right) \, , \tag{4.28}$$

is invertible and is the agent responsible for "changing" holonomic (external, spacetime) indexes to nonholonomic (internal, gauge) ones. This is the geometrical interpretation of the field $e^a{}_\mu(x)$ introduced by Kibble when gauging the Poincaré group.

When a Lorentzian metric exists on P, we can always choose to trace a section that selects only orthogonal frames in relation to this metric. Remember that a change of section is nothing but a change of gauge. These frames will transform accordingly to the Lorentz subgroup of $GL(4, \mathbb{R})$ instead. This actually represents a bundle contraction $GL(4, \mathbb{R}) \rightarrow SO(1, 3)$ and it is this contracted bundle the principle frame bundle underlying the ECSK theory.

In the ECSK approach, A is a connection on the contracted $SO(1, 3)$-bundle of linear frames and e is a soldering form. They should be regarded as independent fields - here Cartan's philosophy starts to emerge. In this contexts, the spacetime metric $g_{\mu\nu}$ and the affine connection $\Gamma^\alpha{}_{\mu\nu}$ are just composite field

$$g_{\mu\nu}(x) = e^a{}_\mu(x)e^b{}_\nu(x)\eta_{ab} \, , \tag{4.29a}$$

$$\Gamma^\alpha{}_{\mu\nu}(x) = e_a{}^\alpha \left(A^a{}_{b\mu}e^b{}_\nu + \partial_\mu e^a{}_\nu\right) \, , \tag{4.29b}$$

where $e_a{}^\mu$ is the inverse soldering form.

The curvature F of A is actually related to the spacetime curvature tensor $R^\alpha{}_{\beta\mu\nu}$,

$$F = \frac{1}{2}g^{\lambda\beta}R^\alpha{}_{\beta\mu\nu}e^a{}_\alpha e^b{}_\lambda \left(\sigma_{ab} \otimes dx^\mu dx^\nu\right) \, . \tag{4.30}$$

The existence of the soldering e - absent in traditional gauge theories - allow us to define the so-called torsion 2-form

$$T \equiv De \, , \tag{4.31}$$

which is related to the spacetime torsion tensor $T^\alpha{}_{\mu\nu}$,

$$T = \frac{1}{2} T^\alpha{}_{\mu\nu} e^a{}_\alpha \left(\theta_a \otimes dx^\mu dx^\nu\right) \; . \tag{4.32}$$

Finally, the curvature F as well as the torsion T satisfy Bianchi identities

$$DF = 0 \; , \tag{4.33a}$$

$$DT = [F, e] \; . \tag{4.33b}$$

In this gauge theoretical formulation the generalized Einstein-Hilbert action (4.14) can be written as

$$S_{EH}[e, A] = \frac{1}{32\pi G} \int \mathrm{Tr} \left[F \star (ee) - \frac{\Lambda^2}{12} ee \star (ee) \right] \; , \tag{4.34}$$

in which the EH term looks suspiciously YM-like, except from the fact it is not power-counting renormalizable since e has canonical dimension -1 and that it involves only the first derivatives of the fields.

The minimal coupling to matter fields Φ do not naturally follow from this formalism, but has to be imposed as an extra physical principle,

$$S[e, A, \Phi] = S_{EH}[e, A] + \int \mathcal{L}(e, \Phi, D\Phi) \; . \tag{4.35}$$

Finally, the Euler-Lagrange equations can be obtained by varying (4.35) independently with respect to e and A. Respectively, we have

$$\left[F^\star - \frac{\Lambda^2}{12} (ee)^\star, e \right] = -32\pi G \tilde{\tau} \; , \tag{4.36a}$$

$$\left[T^\star, e \right] = -32\pi G s \; , \tag{4.36b}$$

where *, remember, is the "color" Lie dual, $\tilde{\tau}$ is the energy-momentum 3-form related to the modified energy-momentum tensor of Einstein-Cartan theory,

$$\tilde{\tau} \equiv \frac{\delta \mathcal{L}}{\delta e} = e_a{}^\mu \tilde{\tau}_{\mu\nu} \left(\theta^a \otimes \star dx^\nu\right) \; , \tag{4.37}$$

and s is the spin density 3-form related to the spin density tensor,

$$s \equiv \frac{\delta \mathcal{L}}{\delta A} = e^a{}_\alpha e_b{}^\mu s^\alpha{}_{\mu\nu} \left(\sigma_a{}^b \otimes \star dx^\nu\right) \; . \tag{4.38}$$

Field equations (4.36), again, display the first order feature. This is actually a feature very characteristic of the Scima-Kibble approach. For this reason, this gauge theoretical approach is sometimes refered to as the first order formalism for gravity.

When expanded in a basis, (4.36) exactly reproduce the Einstein-Cartan field equations (4.16). In other words, the ECSK theory is, as already mentioned, the Einstein-Cartan theory written in the gauge theoretical framework of fiber bundles.

4.3.3   The Lovelock-Cartan-Sciama-Kibble theory

The Lovelock-Cartan-Sciama-Kibble (LCSK) theory of gravity consists of the most general $SO(1,3)$ gauge theories of gravity for Riemann-Cartan spacetimes. It can be seen as a generalization of the ECSK theory, in the sense that it includes all possible coupling that can be add to the action functional without spoiling its first order feature as well as locality and polynomiality.

In a 4-manifolds it reads

$$S\left[e, A\right] = \int \text{Tr} \left[\alpha_1 FF + \alpha_2 FF^{\star} + \alpha_3 F \star (ee) + \alpha_4 ee \star (ee) + \alpha_5 Fee\right] , \qquad (4.39)$$

where α_1 and α_2 have vanishing canonical dimension, α_3 has dimensions -2, α_4 -4 and α_5 -2. This was first obtained by J. Zanelli and A. Mardones in the early nineties [140].

The α_1 and α_2 couplings are recognizable as the Pontryagin and Gauss-Bonnet density, respectively, and, again, we stress that they are topological in nature. The α_3 coupling is the EH term. In particular, $\alpha_3 = (32\pi G)^{-1}$ by the Correspondence Principle. The g_4 coupling is usually associated to the cosmological constant, $\alpha_4 = (-\Lambda^2/384\pi G)$. Finally, g_5 coupling is novel; associated to the torsion tensor T,

$$d \, \text{Tr}\left(e T\right) = \text{Tr}\left(TT - Fee\right) , \qquad (4.40)$$

where $d \, \text{Tr}(e T)$ is the Nieh-Yang density.

The vacuum field equations are

$$\left[\alpha_3 F^{\star} - 2\alpha_4 (ee)^{\star} + \alpha_5 F, e\right] = 0 , \qquad (4.41a)$$

$$\left[\alpha_3 T^{\star} + \alpha_5 T, e\right] = 0 . \qquad (4.41b)$$

Notice that, differently from (4.16b), (4.41b) is not algebraic for the T anymore. In other words, the LCSK theory has a propagating torsion field thus its dynamics vastly differs from the ECSK counterpart and thus from GR. Particular Riemannian solutions, of course, can be obtained

by setting T = 0.

$$F = \Lambda^2 ee \; , \tag{4.42a}$$

$$T = 0 \; , \tag{4.42b}$$

for instance, is the de Sitter spacetime.

4.3.4   Advantages

The ECSK as well as LCSK theories, of course, shares the same advantages that the Einstein-Cartan theory has over GR (less background, avoidance of singularities, etc). Moreover, these theories are formulated in frames that are completely independent of coordinate systems - the so-called nonholonomic frames. This means that this formalism is generally covariant by design, which makes explict the fact that invariance under general coordinate transformations is a physically trivial requirement.

Being formulated in the same mathematical language of YM theories, that of fiber bundles, the Sciama-Kibble approach presents the most appropriated formalism for which to compare gravity to traditional gauge theories. This is particularly important when one wishes to understand perturbative QG, specially its relation to topological gauge theories in dimensions lower than four.

Clearly, what makes gravity so peculiar is the soldering form. By soldering the gauge space to spacetime we are left with no alternative but to consider disconnected and/or noncompact gauge groups such as $GL(4, \mathbb{R})$, $ISO(1, 3)$, $SO(1, 3)$, $\mathbb{R}^4$, etc. Not only that, but its nonvanishing canonical dimension jeopardizes the power-counting renormalizability of gravity.

Finally, the gauge symmetry is one of the most powerful symmetry in Theoretical Physics. As we saw in chapter 2, it translates to very strong set of Ward identities in the quantum theory that prevents a lot of pathologies that a QFT might have otherwise.

The LCSK theory is particularly advantageous in the issue of perturbative renormalizability since it considers more couplings allowed by the gauge symmetry. Of course, YM-like couplings such as $F \star F$ and $T \star T$ are still missing and will probably appear in the counterterm. These are considered as high derivative terms and usually lead to issues in the unitarity. In despite of these problems, the gauge theoretical approach is, if any, the most compelling framework to develop a consistent pertubative QFT for gravity.

Chapter 5

Renormalizable TQFT for gravity

5.1 Introduction

Witten, in his seminal 1988 papers [68, 68, 67] about TQFTs, raised the intriguing possibility that such theories could perhaps describe a symmetry-restored phase of gravity. This possibility has been explored him and many authors with particular success in lower spacetime dimensions.

For instance, $d = 2$ models of topological gravity were studied in [68, 103, 141–145] as well as their coupling to topological σ-models and relation to Matrix Models and topological strings. In $d = 3$, gravity turns out to be perturbatively related to Chern-Simons theory [146, 104, 147, 148]. And, in $d = 4$, topological models of gravity were proposed in [149–152]. A more general discussion about the relation of TQFTs and QG can be found in [153].

Here, we will propose a renormalizable TYM that can generate gravity if its topological symmetry is explicitly broken.

5.2 Unbroken phase

Let us consider the most general local, polynomial and power-counting renormalizable topological theory with gauge group $SO(1, 3)$ on a 4-manifold. Its action functional is given by

$$S_0[A] = \int \text{Tr} \left(g_1 FF + g_2 FF^\star \right) , \tag{5.1}$$

which is clearly TYM-like.

The first term in (5.1) is Pontryagin invariant and the second term is Euler's topological invariant. Both of the coupling parameters g_1 and g_2 are dimensionless. The gauge symmetry

enjoyed by (5.1) is, of course, the same as TYM's, namely,

$$\delta A = -D\xi + \zeta \,, \tag{5.2a}$$

$$\delta\zeta = -D\lambda \,, \tag{5.2b}$$

$$\delta F = -D\zeta - [\zeta, F] \,, \tag{5.2c}$$

where ξ is the usual algebra-valued 0-form gauge parameter; ζ is algebra-valued 1-form gauge parameter which characterizes all other possible transformations that leave (5.1) invariant and λ is also a algebra-valued 0-form gauge parameter which describes redundancies of ζ.

In the BRST technology, the infinitesial transformations is promoted to the nilponent BRST operator and each gauge parameter is promoted to a ghost field. The ghost fields themselves transform, resulting in the set of topological BRST transformations

$$sA = -Dc + \psi \,, \tag{5.3a}$$

$$s\psi = -D\phi - [c, \psi] \,, \tag{5.3b}$$

$$sc = -cc + \phi \,, \tag{5.3c}$$

$$s\phi = - [c, \phi] \,, \tag{5.3d}$$

$$sF = -D\psi - [c, F] \,. \tag{5.3e}$$

To remind the reader, s is the nilpotent topological BRST operator; c is the FP ghost 0-form; ψ is the topological ghost 1-form and; ϕ is the topological ghost 0-form. The grading of these fields can be found in Table 5.1.

5.2.1   Adding a s-exact term

As reviewed in chapter 3, the observables of a theory will not be altered by the addition of s-exact terms to the original action. Observables live in the cohomological groups of the BRST operator s while s-exact terms do not. Moreover, the doublet theorem ensures that fields in a BRST doublet structure cannot be present in gauge invariant quantities.

Let us introduce a pair of field Y and X in a BRST doublet structure

$$sY = X \,, \tag{5.4a}$$

$$sX = 0 \,, \tag{5.4b}$$

whose gradings are also displayed in Table 5.1, and consider the most general action functional that, again, is local, power-counting renormalizable and polynomial, which incorpotares Y

and X in a BRST trivial fashion. This is

$$
\begin{aligned}
S_{\mathrm{triv}} &= s \int \mathrm{Tr}\left[Y\left(g_3 F + g_4 \star F + g_5 F^\star + g_6 \star F^\star + g_7 X + g_8 \star X + g_9 X^\star + g_{10} \star X^\star \right)\right] , \\
&= \int \mathrm{Tr}\left\{ \left(g_3 F + g_4 F \star + g_5 F^\star + g_6 F^\star \star + g_7 X + g_8 X \star + g_9 X^\star + g_{10} X^\star \star \right) X + \right. \\
&\quad + Y\left[g_3 \left(D\psi + [c, F]\right) + g_4 \star \left(D\psi + [c, F]\right) + g_5 \left(D\psi + [c, F]\right)^\star + \right. \\
&\quad \left.\left. + g_6 \star \left(D\psi + [c, F]\right)^\star \right]\right\} ,
\end{aligned}
\tag{5.5}
$$

where, remember, $\star$ is the spacetime Hodge dual while * is the "color" Hodge dual or Lie dual, for short. We remark that the presence of the Hodge dual in S_{triv} does not spoil the topological nature of our theory since it is introduced within a s-exact term. In other words, the BRST symmetry prevents such contamination.

Finally, the full action to be considered is

$$
S = S_0 + S_{\mathrm{triv}} ,
\tag{5.6}
$$

which, of course, is physcally indistinguishable to the theory defined by S_0 alone. After all, they share the same set of physical observables.

5.2.2   Observables

The topological BRST operator in consideration is the virtually the same as the one of the traditional TYM theory introduced in chapter 3. Thefore, this operator defines the same cohomological groups over the moduli space. In other words, the theory defined by action (5.6) also has Donalsdon invariants as its set of physical observables.

The scenario we ask the reader to consider in this chapter thus consists of $d = 4$ QG being fundamentally described by such TQFT. The reason for such consideration, i.e., the relation of such TYM theory to gravity, will be clarified in the following sections. The point here is that, if we assume such scenario, the observables of $d = 4$ QG are, of course, Donaldson invariants: polynomials topological in nature classifying the different smooth structures one may have over the spacetime manifold.

We stress the paradigm shift: there is no propagation of light signals, no gravitions, or local degrees of freedom whatsoever. In such approach to $d = 4$ QG, our knowledge is restricted to the global dynamics and features of the "quantum spacetime".

5.3   Broken phase

A dynamical theory of gravity can be obtained from the action (5.6) if sources Y and X attain the physical values

$$Y\big|_{\text{phys.}} = 0 \,, \tag{5.7a}$$

$$X\big|_{\text{phys.}} = \mu^2 ee \,, \tag{5.8a}$$

where μ is a mass parameter and e is the vierbein field. Indeed, if such, action (5.6) becomes

$$S\big|_{\text{phys.}} = \int \text{Tr} \left\{ g_1 FF + g_2 FF^\star + \mu^2 \left[(g_4 + g_5)F \star (ee) + \mu^2 (g_8 + g_9)ee \star (ee) + \right. \right.$$
$$\left. \left. + (g_3 + g_6)\, Fee \right] \right\} \,, \tag{5.9}$$

which can be immediately recognized as the Lovelock-Cartan theory of gravity on a 4-manifold - *vide* (4.39).

In particular,

$$g_1 = \alpha_1 \,, \tag{5.10a}$$

$$g_2 = \alpha_2 \,, \tag{5.10b}$$

$$\mu^2 (g_4 + g_5) = \alpha_3 \,, \tag{5.10c}$$

$$\mu^4 (g_8 + g_9) = \alpha_4 \,, \tag{5.10d}$$

$$\mu^2 (g_3 + g_6) = \alpha_5 \,. \tag{5.10e}$$

where α_i's are the gravitational coupling parameters in (4.39). It is interesting to point out to the reader that the traditional QFT toolbox, such as the renormalization group equation, could be used here to evaluate the behavior of the renormalized couplings g_i's and thus predict the effective values of α_i's, i.e., of the classical LCSK gravitational couplings. Particular importance should be given to α_3, related to Newton's constant and α_4, related to the cosmological constant.

5.3.1   Symmetry breaking

The topological BRST operator s can be splitted into two pieces

$$s = s_{\text{YM}} + s_{\text{T}} \tag{5.11}$$

where s_{YM} is the traditional YM BRST and s_T the topological shift part. In particular, they satisfy

$$s_{YM}{}^2 = 0 \, , \tag{5.12a}$$

$$[s_{YM}, s_T] = 0 \, , \tag{5.12b}$$

and act on the fields, individually, accordingly to

$$s_{YM}A = -Dc \, , \qquad s_T A = \psi \, , \tag{5.13a}$$

$$s_{YM}c = -cc \, , \qquad s_T c = \phi \, , \tag{5.13b}$$

$$s_{YM}\psi = -[c, \psi] \, , \qquad s_T \psi = -D\phi \, , \tag{5.13c}$$

$$s_{YM}\phi = -[c, \phi] \, , \qquad s_T \phi = 0 \, , \tag{5.13d}$$

$$s_{YM}F = -[c, F] \, , \qquad s_T F = -D\psi \, . \tag{5.13e}$$

Clearly, s_T is nilpotent up to a gauge transformation. For instance,

$$s_T{}^2 A = -D\phi \, . \tag{5.14}$$

Thus, it only defines a cohomology in the space of gauge invariant objects - which represents no obstruction whatsoever since this is the space where observables live. Such peculiar cohomology is known as the "equivariant" cohomology. Such BRST operator was the one obtained by Witten when twisting $\mathcal{N} = 2$ Super-YM theory. In particular, he used this special cohomology to construct the Donaldson invariants from his twisted YM-like theory [69, 154].

Equations (5.7), on the other hand, actually represent a explict break of s_T while s_{YM} is kept intact. In other words,

$$s_T S\big|_{\text{phys.}} \neq 0 \tag{5.15}$$

while

$$s_{YM}S\big|_{\text{phys.}} = 0 \, . \tag{5.16}$$

Thus, as promised, the LCSK theory can be generated by (5.6) via a (partial) breaking of its topological BRST symmetry.

The actual mechanism responsible for this breaking will be left for a future work. Nonetheless, this issue has already being tackled by some authors. For instance, in [155] the breaking occurs due to cotributions coming from the coupling to a "topological matter" sector and in [151, 156] it occurs due to a Higgs-like mechanism.

5.4   Perturbative renormalizability

Here we will employ again the algebraic renormalizability technique to prove that such TYM theory for gravity, defined by action (5.6), is stable under quantum corrections to all orders in perturbation theory.

We will follow the same steps of chapter 3, where the renormalizability of the usual TYM theory was worked out in the (anti-)self-dual Landau gauge. For the convenience of the reader, we repeat the main equations that will be used.

The gauge constraints are

$$d \star A = 0 \, , \tag{5.17a}$$

$$F \pm \star F = 0 \, , \tag{5.17b}$$

$$d \star \psi = 0 \, , \tag{5.17c}$$

which will be implement to the action with the helps of the BRST doublets

$$s\bar{c} = b \, , \quad sb = 0 \, , \tag{5.18a}$$

$$s\bar{\chi} = B \, , \quad sB = 0 \, , \tag{5.18b}$$

$$s\bar{\phi} = \bar{\eta} \, , \quad s\bar{\eta} = 0 \, . \tag{5.18c}$$

The gauge fixing action is then

$$
\begin{aligned}
S_{\text{gf}} = s \int \text{Tr} \left[\bar{c}d \star A + \bar{\chi} \left(F \pm \star F \right) + \bar{\phi}d \star \psi \right] \, , \\
= \int \text{Tr} \left\{ bd \star A - \bar{c}d \star Dc + \left(\bar{c} + \bar{\eta} + \left[c, \bar{\phi} \right] \right) d \star \psi + \bar{\phi}d \star D\phi + \right. \\
\left. + \, dc \left[\star\psi, \bar{\phi} \right] + \left(B + \left[\bar{\chi}, c \right] \right) \left(F \pm \star F \right) + \bar{\chi} \left(D \pm \star D \right) \psi \right\} \, .
\end{aligned}
\tag{5.19}
$$

The nonlinearities of the symmetries enjoyed by the theory will be introduced to the action with the help of the BRST dublets

$$s\tau = \Omega \, , \quad s\Omega = 0 \, , \tag{5.20a}$$

$$sE = L \, , \quad sL = 0 \, , \tag{5.20b}$$

The external action is then

$$S_{\text{ext}} = s \int \text{Tr} \left(\tau Dc + Ecc \right) ,$$
$$= \int \text{Tr} \left[\Omega Dc + \tau \left(D\phi + [c, \psi] \right) + Lcc + E \left[c, \phi \right] \right] \tag{5.21}$$

The gradings of all these fields are displayed in table 5.1.

Finally, the total action of interest is given by

$$\Sigma = S + S_{\text{gf}} + S_{\text{ext}} \tag{5.22}$$

where S is given by (5.6), of course.

Table 5.1 Gradings of the fields as differential forms on M and on $\mathcal{M}$.

Fields	A	c	ψ	ϕ	Y	X	$\bar{c}$	b	$\bar{\chi}$	B	$\bar{\phi}$	$\bar{\eta}$	τ	Ω	E	L
M	1	0	1	0	2	2	0	0	2	2	0	0	3	3	4	4
$\mathcal{M}$	0	1	1	2	-1	0	-1	0	-1	0	-2	-1	-2	-1	-3	-2

5.4.1   Ward identities

This set of Ward identities will differ slightly from the one of traditional TYM due to the presence of S_{triv}. This, however, will not jeopardize the renormalizability.

We start with the traditional gauge fixing equation

$$\frac{\delta \Gamma}{\delta b} = d \star A , \tag{5.23}$$

followed by the FP antighost equation

$$\mathcal{G}_{\bar{c}} \left(\Gamma \right) = d \star \psi , \tag{5.24}$$

the topological ghost gauge fixing equation,

$$\frac{\delta \Gamma}{\delta \bar{\eta}} = d \star \psi , \tag{5.25}$$

the bosonic antighost equation,

$$\mathcal{G}_{\bar{\phi}} \left(\Gamma \right) = 0 , \tag{5.26}$$

and Slavnov-Taylor identity

$$\mathcal{S}(\Gamma) = 0 , \tag{5.27}$$

where the Slavnov-Taylor operator is now given by

$$\mathcal{S} \equiv \int \text{Tr} \left[\left(\psi - \frac{\delta}{\delta\Omega} \right) \frac{\delta}{\delta A} + \left(\phi - \frac{\delta}{\delta L} \right) \frac{\delta}{\delta c} - \frac{\delta}{\delta\tau} \frac{\delta}{\delta\psi} - \frac{\delta}{\delta E} \frac{\delta}{\delta\phi} + b\frac{\delta}{\delta\bar{c}} + B\frac{\delta}{\delta\bar{\chi}} + \right.$$
$$\left. + +\bar{\eta}\frac{\delta}{\delta\bar{\phi}} + \Omega\frac{\delta}{\delta\tau} + L\frac{\delta}{\delta E} + X\frac{\delta}{\delta Y} \right] , \tag{5.28}$$

and its linearized version by

$$\mathcal{S}_\Gamma \equiv \int \text{Tr} \left[\left(\psi - \frac{\delta\Gamma}{\delta\Omega} \right) \frac{\delta}{\delta A} - \frac{\delta\Gamma}{\delta A}\frac{\delta}{\delta\Omega} + \left(\phi - \frac{\delta\Gamma}{\delta L} \right) \frac{\delta}{\delta c} - \frac{\delta\Gamma}{\delta c}\frac{\delta}{\delta L} - \frac{\delta\Gamma}{\delta\tau}\frac{\delta}{\delta\psi} - \frac{\delta\Gamma}{\delta\psi}\frac{\delta}{\delta\tau} + \right.$$
$$\left. - \frac{\delta\Gamma}{\delta E}\frac{\delta}{\delta\phi} - \frac{\delta\Gamma}{\delta\phi}\frac{\delta}{\delta E} + b\frac{\delta}{\delta\bar{c}} + B\frac{\delta}{\delta\bar{\chi}} + \bar{\eta}\frac{\delta}{\delta\bar{\phi}} + \Omega\frac{\delta}{\delta\tau} + L\frac{\delta}{\delta E} + X\frac{\delta}{\delta Y} \right] . \tag{5.29}$$

Finally, we also have the topological ghost equation

$$\mathcal{G}_\phi(\Gamma) = \Delta_\phi , \tag{5.30}$$

where the breaking is now given by

$$\Delta_\phi \equiv \int ([A, \tau] + [c, E]) , \tag{5.31}$$

the first FP ghost equation

$$\mathcal{G}_c^1(\Gamma) = \Delta_c , \tag{5.32}$$

where it is now given by

$$\mathcal{G}_c^1 \equiv \int \left(\frac{\delta}{\delta c} - \left[\bar{c}, \frac{\delta}{\delta b} \right] - \left[\bar{\chi}, \frac{\delta}{\delta B} \right] - \left[\bar{\phi}, \frac{\delta}{\delta\bar{\eta}} \right] + \left[Y, \frac{\delta}{\delta X} \right] \right) , \tag{5.33}$$

and the breaking by

$$\Delta_c \equiv \int ([A, \Omega] + [\tau, \psi] + [c, L] + [E, \phi]) , \tag{5.34}$$

and the second FP ghost equation

$$\mathcal{G}_c^2(\Gamma) = \Delta_c , \tag{5.35}$$

where

$$\mathcal{G}_c^2 \equiv \int \left(\frac{\delta}{\delta c} - \left[\bar{\phi}, \frac{\delta}{\delta \bar{c}} \right] + \left[A, \frac{\delta}{\delta \psi} \right] + \left[c, \frac{\delta}{\delta \phi} \right] + \left[\bar{\eta}, \frac{\delta}{\delta b} \right] + \left[E, \frac{\delta}{\delta L} \right] + \left[\tau, \frac{\delta}{\delta \Omega} \right] \right) . \tag{5.36}$$

5.4.2 Counterterms

Considering the quantum action at 1-loop

$$\Gamma^{(1)} = \Sigma + \epsilon \Sigma^{\text{ct}} , \tag{5.37}$$

the set of Ward identities resume to

$$\frac{\delta \Sigma^{\text{ct}}}{\delta b} = 0 , \tag{5.38a}$$

$$\mathcal{G}_{\bar{c}} \left(\Sigma^{\text{ct}} \right) = 0 , \tag{5.38b}$$

$$\frac{\delta \Sigma^{\text{ct}}}{\delta \bar{\eta}} = 0 , \tag{5.38c}$$

$$\mathcal{G}_{\bar{\phi}} \left(\Sigma^{\text{ct}} \right) = 0 , \tag{5.38d}$$

$$\mathcal{S}_{\Sigma} \left(\Sigma^{\text{ct}} \right) = 0 , \tag{5.38e}$$

$$\mathcal{G}_{\phi} \left(\Sigma^{\text{ct}} \right) = 0 , \tag{5.38f}$$

$$\mathcal{G}_c \left(\Sigma^{\text{ct}} \right) = 0 . \tag{5.38g}$$

Its solution

$$\Sigma^{\text{ct}} = \mathcal{S}_{\Sigma} \int \text{Tr} \left[a_1 \left(\Omega A + A \star d\bar{c} + \tau \psi + \psi \star d\bar{\phi} \right) + a_2 \left(\tau dc + \bar{\phi} d \star dc \right) + \right.$$
$$\left. + \; a_3 \bar{\chi} d A + a_4 \bar{\chi} A A \right] \tag{5.39}$$

is the most general counterterm allowed.

5.4.3 Quantum stability

Finally, one can show that Σ^{ct} can be absorbed in Σ by a redefinition

$$\Phi_0 \equiv z_{\Phi} \Phi , \tag{5.40a}$$

$$g_0 \equiv z_g g , \tag{5.40b}$$

$$\mathcal{J}_0 \equiv z_{\mathcal{J}} \mathcal{J} , \tag{5.40c}$$

of fields $\Phi \in \left\{ A, c, \psi, \phi, Y, X, \bar{c}, b, \bar{\chi}, B, \bar{\phi}, \bar{\eta} \right\}$, parameter g and external sources $\mathcal{J} \in \{\tau, \Omega, E, L\}$. In other words, that

$$\Sigma \left[\Phi, g, \mathcal{J} \right] + \epsilon \Sigma^{ct} \left[\Phi, g, \mathcal{J} \right] = \Sigma \left[\Phi_0, g_0, \mathcal{J}_0 \right] \ . \tag{5.41}$$

In particular, the nontrivial z-factors can be evaluated as

$$z_{\mathrm{A}} z_{\mathrm{B}} = z_{\bar{\chi}} z_c = z_{\bar{\chi}} z_\psi = 1 + \epsilon a_4 \ , \tag{5.42a}$$

$$z_g z_c z_{\bar{\chi}} = z_g z_\phi z_\chi = z_g^2 z_\tau z_c z_\psi = z_g z_{\mathrm{B}} z_c z_\phi = 1 + \epsilon a_2 \ , \tag{5.42b}$$

$$z_\Omega z_c = z_\tau z_\phi = z_g z_\tau z_c z_\psi = z_g z_{\mathrm{L}} z_c = z_g z_{\mathrm{E}} z_c z_\phi = 1 + \epsilon a_3 \ , \tag{5.42c}$$

$$z_{\mathrm{A}} z_g = z_c z_{\bar{c}} = z_b z_{\mathrm{A}} = z_{\bar{\eta}} z_\psi = z_{\bar{c}} z_\psi = z_g z_{\bar{\phi}} z_c z_\psi = 1 + \epsilon a_1 \ . \tag{5.42d}$$

Chapter 6

Conclusions and perspectives

Witten's hypothesis of a $d = 4$ TQFT describing an unbroken phase of gravity motivated this thesis. In order to propose a novel scenario that implements such idea in a consistent way, a detailed study of the renormalizability properties of TYM theories in the (anti-)self-dual Landau gauge was made by the author and collaborators. In chapter 3, a review of TYM theory was made and the main results of such efforts were highlighted.

TYM theory have a particularly strong set of Ward identities. As shown in chapter 3, it renormalizes with only one (unphysical) parameter [109]. This is an improvement over previous results that were avaliable in the literature [110, 21, 22].

The propagator $\langle A(x)A(y) \rangle$, in particular, vanishes exactly [109]. This is due to the presence of the vectorial supersymmetry $\mathcal{W}$. As a consequence, TYM theory in the (anti-)self-dual Laudau gauge is tree-level exact [111]. In other words, it suffers no radiative corrections whatsoever and the path integral is exactly soluable in the semi-classical approximation.

TQFTs in general are defined as theories whose observables are global invariants and do not depend on the metric structure of spacetime. They are generally covariant and diffeomorphism invariant by design and cannot be formulated otherwise. In this sense, they are background independent, providing us information about the topology or smooth structure of spacetime.

These features are indeed very appealing for a quantum theory of gravity. Hence Witten's speculation. In chapter 5, a proposal of QG model was made which consists of a TYM that can generate the family of LCSK gravities. In particular, the local degrees of freedom of gravity are enprisioned by the topological BRST symmetry and are only unleashed after its explicit breaking.

The breaking mechanism was not worked out in this thesis and will be left for future investigation. Some proposals have been made concerning this issue. In [155], for instance, the breaking can happen due to the coupling to a topological matter sector. In [151] and [157], it happens via a Higgs-like mechanism. In particular, it is of most importance that it

happens in the vicinity of Planck energy scale. In this way, the topological description would be restricted to the trans-Planckian regime. Then again, we can only speculate about that at the present moment.

We have shown that this particular TYM theory is physically consistent. At least in the sense that it is renormalizable to all orders. The observables can be evaluate as the Donaldson's invariants, classifying the smooth strucute of spacetime. A consistent scenario then builds, of a perturbative QFT describing the "quantum structure" of spacetime via its global invariants.

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